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ON THE THEORY OF THE ELECTRON WAVE TUBE WITH ELLIPTIC CROSS SECTION

BY

PENTTI MATTILA

HELSINKI 1957

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Preface

In 1950, I had opportunity to pursue studies at Chalmers University of Technology, Gothenburg, by virtue of a scholarship for young practisers of science from Finland. I remember with particular gratitude my teachers of that time, Professor O. E. H. RYDBECK and Professor HENRY WALLMAN. Under the guidance of Prof. RYDBECK, I participated in research work concerning electron wave tubes. Also, I attended the exacting classes of Prof. RYDBECK. Prof. WALLMAN, again, gave an inspiring series of lectures for graduate students on applied mathematics. Furthermore, these lectures gave me numerous impulses toward my future studies.

After this, I went to study at Princeton University. Among many courses, I took there, I remember especially Professor HAMILTON's advanced one in electromagnetic theory. Now I felt inclined to use the knowledge I had acquired and to extend the theory of electron wave tubes to elliptic cross sections. It was readily evident from a physical point of view that, although even quite difficult equations might be encountered in the course of the work, they must certainly have real solutions. My work was also particularly facilitated by the fact that I had become familiar with the theory of circular cross section electron wave tubes under the guidance of Prof. RYDBECK. The attentive reader will easily notice that I have made use of this theory in numerous instances.

In this manner, the work in hand has come into being. It was completed already in 1951. Several circumstances have delayed its publication so that it is available in print only now, after more than five years.

I have the pleasant duty to acknowledge the help of Professor P. JAUHO, who has read the entire manuscript and made numerous noteworthy comments at its finishing stage.

Finally I wish to thank Mr. U. ATTILA, M. SC., for his help in the translation of this work.

Helsinki, January 31, 1957

The Author

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Introduction

The purpose of this paper is to derive the theory of the electron wave tube of elliptic cross section. The theory presented in the following is based on small signal theory, equivalent to a simplification of the equations relating to the electron flow by neglecting quantities which become small when the signal is small. Furthermore, the magnetic field acting longitudinally in the tube is assumed to be of such magnitude that one is justified in neglecting the transversal motion of the electrons, the focussing forces of the field attaining adequate magnitude to prevent this kind of motion.

In electron wave tubes amplification is obtained in the electron flow, in which two or more electron streams of different velocities are mixed. If the densities of charge of the streams are sufficiently high, interaction occurs between the streams, making possible an amplification of the modulating wave in the velocity-modulated flow. It should be noted that the amplification occurs in the electron flow itself. In this way, thus, a travelling wave with negative damping is produced. Modulation is suitably imparted to the electron flow by letting it pass, for instance, through a resonator excited by the input signal. The amplified output signal is similarly obtained by letting the electron flow act, for instance, upon a resonator tuned to the appropriate frequency. It is found that the elctron wave tube is suitable for very short wavelengths. The gain is essentially dependent on the distance between input point and output point. It is thus possible in principle to obtain very high gain by this method, if the length of the tube is sufficiently increased.

In the present paper the particular case will be considered in which two electron streams with different velocities and D. C. densities travel in one and the same electron flow. The wave proceeds in the direction of movement of the electrons. The aim of this work is to calculate the gain in the elliptic wave tube described above and to compare the results to those which apply to the wave tube of circular cross section.

Principle

The electron wave tube consists in principle of a metal tube, inside of which a strong axial magnetic field exists. Two or more electron streams

travel with different velocities within the tube, and are either completely mixed or concentrically separate. The tube is under vacuum. Electromagnetic waves proceeding in such a tube can be divided into two parts, the TM and TE waves. It is found that only the TM waves possess an interaction region, and a study of the gain in the electron wave tube can be restricted to concern the properties of the TM waves. Before an analytic expression can be written for the TM waves, the dynamics of the electron stream have to be studied and the axial current density has to be determined in terms of the axial electric field. It is found that in the case of the circular wave tube the solution of the TM wave contains Bessel's functions and in the case of the elliptical tube Mathieu's functions.

In the derivation of theoretical expressions, boundary value considerations between the electron flow and the free space, and between the free space and the metal surface of the tube, will gain remarkable importance. The basic idea is that the tangential components have to be continuous. This actually implies that the impedance has to be the same on either side of the boundary surface. Consequently, no power reflexions occur when a power flow in the direction of the normal upon the interface is concerned. In the case of a conductive wall, this condition implies that the component of the electric field parallel to the boundary surface must vanish, since the conductivity of the metal surface is assumed to be infinite.

Combining the boundary conditions obtained in this manner, one arrives at the principal equation, the solution of which gives the propagation constant. In the elliptic tube the wave is split up into an odd and even wave. Therefore, boundary conditions are also obtained for odd as well as for even waves. It is found, however, that the obtained equations are analogous. It is thus only necessary to treat one of these two cases. The solution leads to the determination of the axial propagation constant, and it is found that an infinite number of values is obtained for this constant. This signifies that, theoretically, there is an infinite number of wave forms which can be propagated in a tube of this kind.

When only one electron stream is concerned, it is seen that no effect of amplification can be found. If the formulas are applied to the case that there are two electron streams with different velocities, completely mixed and moreover modulated, it is found that then the possibility of an amplifying effect is present. The criterion of amplification is the damping factor. Amplification always occurs when this factor becomes negative.

After this, the expression of the total current density and total current is examined. The current is composed of two parts: the bunch current, and the coupling current. The former obeys a sinusoidal law, whereas the latter is a hyperbolic function. It should be noted that it is just the coupling current which contains the amplifying factor. A study of the case that only

one of the two streams is modulated shows that the solution is of the same form as before, but in comparison with the preceding case, moreover, a phase shift is obtained.

At the study of elliptic electron wave tubes the assumption is made that the electron streams are completely mixed and travel in one and the same electron flow, which has the shape of an ellipse confocal with the cross section of the tube. From a theoretical point of view it is obviously interesting to examine the case that the electron flow fills the entire cross section. Of further interest is the case in which the electron streams are not mixed but are separate, with confocal ellipses as their cross sections.

Fundamental equations

Our starting point are Maxwell's field equations

$$\begin{cases} \nabla \times \bar{E} + \dfrac{\partial \bar{B}}{\partial t} = 0 \\ \nabla \times \bar{H} - \dfrac{\partial \bar{D}}{\partial t} = \bar{J} \end{cases} \tag{1}$$

The problem is to express these equations in elliptic coordinates. For this purpose, the conventional method is employed, presenting the equations at first in general orthogonal coordinates, divided into scalar components, and noting then which form they assume in elliptic coordinates (Appendix 1). We introduce the notation shown in fig. 1. Let us furthermore assume that

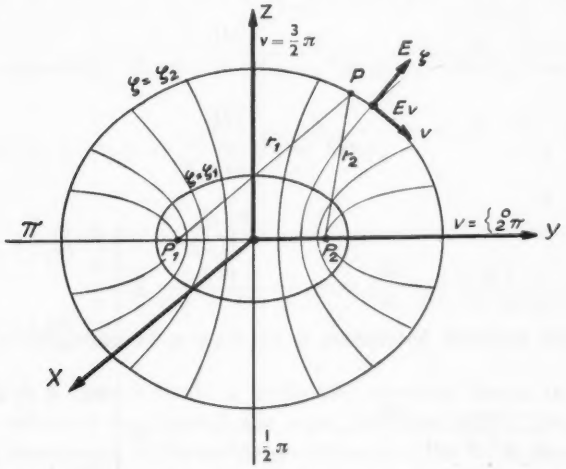


Fig. 1. Elliptic coordinates: x, ζ, v.

the axial propagation is governed by the factor $e^{j\omega t-\gamma x}$. With regard to time, thus, simple harmonic variation is assumed, and the propagation constant γ is defined by means of the expression

$$\gamma = \alpha + j\,\beta \qquad \text{where} \qquad \begin{aligned} \alpha &= \text{damping factor} \\ \beta &= \text{phase factor} \end{aligned}$$

The connection between the rectangular coordinates x, y, z and the elliptic coordinates x, ζ, v is expressed by the transformation formulas

$$\begin{aligned} x &= x \\ y &= q \cosh \zeta \cdot \cos v \\ z &= q \sinh \zeta \cdot \sin v \end{aligned}$$

The surfaces $\zeta =$ constant determine a family of confocal elliptic cylinders, and $v =$ constant a family of confocal hyperbolic cylinders.

With the aid of Appendix 1, it is immediately possible to write the solutions in elliptic coordinates for the TM and TE waves. The solutions are:

TE wave:

$$\left\{\begin{aligned}
H_x &= B \begin{Bmatrix} Se_l\,(c,\cos v)\; Je_l\,(c,\cosh\zeta) \\ So_l\,(c,\cos v)\; Jo_l\,(c,\cosh\zeta) \end{Bmatrix} e^{j\omega t-\gamma x} \\
H_\zeta &= -\frac{\gamma}{j\,\omega\,\mu} E_v = -\frac{\gamma}{q_1\,(k^2+\gamma^2)} \frac{\partial H_x}{\partial \zeta} \\
H_v &= +\frac{\gamma}{j\,\omega\,\mu} E_\zeta = -\frac{\gamma}{q_1\,(k^2+\gamma^2)} \frac{\partial H_x}{\partial v} \\
E_x &= 0 \qquad\qquad (2) \\
E_\zeta &= -\frac{j\,\omega\,\mu}{q_1\,(k^2+\gamma^2)} \frac{\partial H_x}{\partial v} = -\frac{j\,\omega\,\mu}{q_1\,(k^2+\gamma^2)} B \begin{Bmatrix} \dfrac{\partial}{\partial v} Se_l(\,)\; Je_l(\,) \\ \dfrac{\partial}{\partial v} So_l(\,)\; Je_l(\,) \end{Bmatrix} e^{j\omega t-\gamma x} \\
E_v &= +\frac{j\,\omega\,\mu}{q_1\,(k^2+\gamma^2)} \frac{\partial H_x}{\partial \zeta} = +\frac{j\,\omega\,\mu}{q_1\,(k^2+\gamma^2)} B \begin{Bmatrix} Se_l(\,) \dfrac{\partial}{\partial \zeta} Je_l(\,) \\ Se_l(\,) \dfrac{\partial}{\partial \zeta} Jo_l(\,) \end{Bmatrix} e^{j\omega t-\gamma x}
\end{aligned}\right.$$

TM wave:

$$
\begin{cases}
E_x = B \begin{Bmatrix} Se_l\,(c,\cos v)\, Je_l\,(c,\cosh\zeta) \\ So_l\,(c,\cos v)\, Jo_l\,(c,\cosh\zeta) \end{Bmatrix} e^{j\omega t-\gamma x} \\
E_\zeta = +\dfrac{\gamma}{\sigma + j\,\omega\,\varepsilon} H_v = -\dfrac{\gamma}{q_1\,(k^2+\gamma^2)} \cdot \dfrac{\partial E_x}{\partial \zeta} \\
E_v = -\dfrac{\gamma}{\sigma + j\,\omega\,\varepsilon} H_\zeta = -\dfrac{\gamma}{q_1\,(k^2+\gamma^2)} \cdot \dfrac{\partial E_x}{\partial v} \\
H_x = 0 \\
H_\zeta = +\dfrac{\sigma + j\,\omega\,\varepsilon}{q_1\,(k^2+\gamma^2)} \cdot \dfrac{\partial E_x}{\partial v} \\
H_v = -\dfrac{\sigma + j\,\omega\,\varepsilon}{q_1\,(k^2+\gamma^2)} \cdot \dfrac{\partial E_x}{\partial \zeta}
\end{cases}
\tag{3}
$$

It should be noted with regard to the notations employed that k is determined in Appendix 1 by means of the expression

$$k^2 = \omega^2\,\mu\,\varepsilon - j\,\omega\,\mu\,\sigma$$

If particularly a medium with conductivity $\sigma = 0$ is considered, there is

$$k^2 = k_2^2 = \omega^2\,\mu\,\varepsilon = \left(\frac{\omega}{u}\right)^2, \text{ where } u = \frac{1}{\sqrt{\mu\,\varepsilon}}$$

In vacuum, especially, there is $\varepsilon = \varepsilon_0$, $\mu = \mu_0$ and $\sigma = 0$.

Consequently, $u = c_0 = \dfrac{1}{\sqrt{\mu_0\,\varepsilon_0}}$ and thus

$$k^2 = k_0^2 = \left(\frac{\omega}{c_0}\right)^2$$

The propagation of a wave in a modulated electron beam

The problem is now to study a modulated electron beam, in which the electrons are assumed to move in the axial direction only. The motion of the electrons is assumed to contain, in addition to the D. C. component, a small A. C. component caused by the signal. It is supposed to be small

enough in comparison with the D. C. component to justify the use of the theory of small signals. This implies that certain simplifying assumptions are allowed when deriving the A. C. component of the electron velocity. From the velocity of the electrons, again, it is easy to obtain the current density of the electron beam. In this manner a connection is established between the axial current density in the electron beam and the strength of the axial electric field.

We assume that the electrons move only in the x direction. The total values of charge density, current density and velocity of the electron beam are assumed to have the following form:

$$\begin{aligned} \varrho_T &= \varrho_0 + \varrho = \text{charge density} \\ J_T &= J_0 + J_x = \text{current density} \\ v_T &= v_0 + v = \text{velocity} \end{aligned}$$

where

$$\begin{aligned} \varrho &= \text{A. C. component of charge density} \\ \varrho_0 &= \text{mean, or D. C., value of charge density} \\ J_x &= \text{A. C. component of current density} \\ J_0 &= \text{mean, or D. C., value of current density} \\ v &= \text{A. C. component of electron velocity} \\ v_0 &= \text{mean, or D. C., value of electron velocity} \end{aligned}$$

Let us further assume that the variation of the A. C. components is sinusoidal with regard to time and that they are governed by one and the same propagation constant γ in the direction of propagation, x. We may thus write

$$\begin{aligned} \varrho &= Re\ (\hat{\varrho}\ \ e^{j\omega t - \gamma x}) \\ J_x &= Re\ (\hat{J}_x\ e^{j\omega t - \gamma x}) \\ v &= Re\ (\hat{v}\ e^{j\omega t - \gamma x}) \end{aligned}$$

where $\hat{\varrho}$, $\hat{J}_x$ and $\hat{v}$ denote the peak values.

It is found from the equation of continuity that

$$\nabla \cdot \bar{J}_T + \frac{\partial \varrho_T}{\partial t} = 0 \qquad (4)$$

$\nabla \cdot \bar{J}_T$ can be expressed in elliptic coordinates, and it will then have the form [Stratton, p. 49 (79)]:

$$\nabla \cdot \bar{J}_T = \frac{1}{q_1^2}\left\{\frac{\partial}{\partial x}\left[q_1^2\,(J_0 + J_x)\right] + \frac{\partial}{\partial \zeta}\,(q_1\,J_\zeta) + \frac{\partial}{\partial v}\,(q_1\,J_v)\right\} = \frac{\partial J_x}{\partial x}$$

since it was assumed that $J_\zeta = J_v = 0$.

In this way, we obtain

$$\nabla \cdot \bar{J}_T + \frac{\partial \varrho_T}{\partial t} = \frac{\partial J_x}{\partial x} + \frac{\partial \varrho}{\partial t} = 0$$

$$\therefore\ -\gamma \cdot J_x + j\,\omega\,\varrho = 0$$

$$J_x = +\,j\,\frac{\omega}{\gamma} \cdot \varrho$$

It is known on the other hand that under the assumed conditions

$$m\,\frac{dv_T}{dt} = -\,e\,E_x \quad , \quad \text{where} \quad \begin{cases} -\,e = \text{charge of the electron} \\ m = \text{electron mass} \end{cases}$$

$$v_T = v_0 + v = f\,(t, x)$$

Now,

$$\frac{dv_T}{dt} = \frac{\partial v_T}{\partial t} \cdot \frac{dt}{dt} + \frac{\partial v_T}{\partial x} \cdot \frac{dx}{dt} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} \cdot v_T$$

In accordance with the theory of small signals, we make the simplifying assumption $v_T = v_0 + v \cong v_0$

$$\therefore\ \frac{dv_T}{dt} = \frac{\partial v}{\partial t} + \frac{\partial v}{\partial x} \cdot v_0 = j\,\omega\,v - \gamma\,v_0\,v$$

$$= j\,v_0\left(\frac{\omega}{v_0} + j\,\gamma\right) v$$

Hence follows

$$j\,v_0\left(\frac{\omega}{v_0} + j\,\gamma\right) v = -\,\frac{e}{m}\,E_x$$

$$\therefore\ v = \frac{-\,\dfrac{e}{m} \cdot E_x}{j\,v_0\left(\dfrac{\omega}{v_0} + j\,\gamma\right)} = \frac{-\,\dfrac{e}{m} \cdot E_x \cdot k_1}{j\,\omega\,(k_1 + j\,\gamma)}$$

where $k_1 = \dfrac{\omega}{v_0}$

The first approximation for the current density is

$$J_T = J_0 + J_x = v_T \varrho_T = (v_0 + v)(\varrho_0 + \varrho) = \varrho_0 v_0 + \varrho v + \varrho_0 v + \varrho v_0$$

$$\cong \varrho_0 v_0 + \varrho v_0 + \varrho_0 v$$

$$\therefore J_x \cong \varrho v_0 + \varrho_0 v$$

We obtain thus, considering that $\varrho_0 = -\frac{J_0}{v_0}$,

$$J_x = \frac{\gamma}{j\omega} \cdot J_x v_0 - \frac{J_0}{v_0} \cdot \frac{-\frac{e}{m} \cdot E_x \cdot k_1}{j\omega(k_1 + j\gamma)}$$

or

$$J_x \left(1 + j\gamma \frac{v_0}{\omega}\right) = + \frac{J_0 \frac{e}{m} E_x \cdot k_1}{j\omega v_0 (k_1 + j\gamma)}$$

$$\therefore J_x = \frac{\frac{e}{m} E_x k_1^2}{j\omega v_0 (k_1 + j\gamma)^2} J_0 = \frac{k_1 \cdot J_0}{2 j U_0 (k_1 + j\gamma)^2} E_x \tag{5}$$

where U_0 denotes the accelerating D. C. voltage which gives the electron the velocity v_0. Therefore,

$$U_0 = \frac{1}{2e} m v_0^2$$

The equation can still be written in a briefer form if the symbol $z_0 = \frac{U_0}{J_0}$ is introduced.

The final expression of the current density will thus be

$$J_x = -j \frac{1}{2 z_0} \cdot \frac{k_1}{(k_1 + j\gamma)^2} \cdot E_x \tag{6}$$

It is now possible to determine the wave propagated in the modulated electron beam. It is seen that, under the conditions assumed to be fulfilled in this case, it has always an axial component of current density and thus also of electric field. This implies that the wave form is of the TM type. Consequently, the solution for the wave propagated in an elliptic electron beam can now be written.

For this purpose, we write, with the aid of Appendix 1, the field equations for the TM wave in component form. In the electron beam, they are

$$
\begin{cases}
\dfrac{\partial}{\partial \zeta}(q_1 E_v) - \dfrac{\partial}{\partial v}(q_1 E_\zeta) & = 0 \\
\dfrac{1}{q_1}\dfrac{\partial}{\partial v}(E_x) + \gamma E_v + j\omega\mu_0 H_\zeta & = 0 \\
-\gamma E_\zeta - \dfrac{1}{q_1}\dfrac{\partial}{\partial \zeta} E_x + j\omega\mu_0 H_v & = 0 \\
\dfrac{\partial}{\partial \zeta}(q_1 H_v) - \dfrac{\partial}{\partial v}(q_1 H_\zeta) & = q_1^2 (j\omega\varepsilon_0 E_x + J_x) \\
+\gamma H_v & = j\omega\varepsilon_0 E_\zeta \\
-\gamma H_\zeta & = j\omega\varepsilon_0 E_v
\end{cases} \tag{7}
$$

From this, the following differential equation is obtained for E_x:

$$
\frac{\partial^2 E_x}{\partial \zeta^2} + \frac{\partial^2 E_x}{\partial v^2} + q_1^2(\omega^2\mu_0\varepsilon_0 + \gamma^2)\left[1 - \frac{1}{2z_0}\cdot\frac{k_1}{\omega\varepsilon_0}\cdot\frac{1}{(k_1 + j\gamma)^2}\right] E_x = 0
$$

Substituting $Z_0 = \sqrt{\dfrac{\mu_0}{\varepsilon_0}}$ and $c_0 = \dfrac{1}{\sqrt{\mu_0\varepsilon_0}}$

we obtain

$$
\frac{\partial^2 E_x}{\partial \zeta^2} + \frac{\partial^2 E_x}{\partial v^2} + q_1^2(k_0^2 + \gamma^2)\left[1 - \frac{Z_0}{z_0}\cdot\frac{c_0}{2v_0}\cdot\frac{1}{(k_1 + j\gamma)^2}\right] E_x = 0
$$

Let $\Lambda^2 = \left[1 - \dfrac{Z_0}{z_0}\cdot\dfrac{c_0}{2v_0}\cdot\dfrac{1}{(k_1 + j\gamma)^2}\right]$

This gives us

$$
\frac{\partial^2 E_x}{\partial \zeta^2} + \frac{\partial^2 E_x}{\partial v^2} + q^2\Lambda^2(k_0^2 + \gamma^2)(\cosh^2\zeta - \cos^2 v) E_x = 0 \tag{8}
$$

Substituting $E_x = f_1(\zeta)\cdot f_2(v)$, we find

$$\begin{cases} \dfrac{\partial^2 f_1}{\partial \zeta^2} + [q^2 \Lambda^2 (k_0^2 + \gamma^2) \cosh^2 \zeta - b_l] f_1 = 0 \\ \dfrac{\partial^2 f_2}{\partial v^2} + [b_l - q^2 \Lambda^2 (k_0^2 + \gamma^2) \cos^2 v] f_2 = 0 \end{cases} \tag{9}$$

We have thus again arrived at Mathieu's functions. It is now immediately possible to write the solutions for the TM wave.

TM wave in an elliptic electron beam:

$$\begin{cases} E_x = B_1 \begin{Bmatrix} Se_l (c_1, \cos v)\, Je_l (c_1, \cosh \zeta) \\ So_l (c_1, \cos v)\, Jo_l (c_1, \cosh \zeta) \end{Bmatrix} e^{j\omega t - \gamma x} \\ E_\zeta = + \dfrac{\gamma}{j \omega \varepsilon_0} H_v \\ E_v = - \dfrac{\gamma}{j \omega \varepsilon_0} H_\zeta \\ H_x = 0 \\ H_\zeta = + \dfrac{j \omega \varepsilon_0}{q_1 (k_0^2 + \gamma^2)} \cdot \dfrac{\partial E_x}{\partial v} \\ H_v = - \dfrac{j \omega \varepsilon_0}{q_1 (k_0^2 + \gamma^2)} \cdot \dfrac{\partial E_x}{\partial \zeta} \end{cases} \tag{10}$$

where

$$c_1 = q \Lambda (k_0^2 + \gamma^2)^{1/2} = \Lambda \cdot c$$

$$k_0 = \frac{\omega}{c_0} \quad \text{and} \quad c = q (k_0^2 + \gamma^2)^{1/2}$$

$$\Lambda = \left[1 - \frac{1}{2} \cdot \frac{Z_0}{z_0} \cdot \frac{c_0}{v_0} \cdot \frac{1}{(k_1 + j\gamma)^2}\right]^{1/2} = \left[1 - \frac{a^2}{(k_1 + j\gamma)^2}\right]^{1/2}$$

It is obvious that the TE wave is not by any means impossible in the elliptic electron beam, but it is not of any interest in this case since, according to definition, then $E_x = 0$. This wave form thus leaves the influence of the electron beam out of account. In the following, therefore, we shall consider the TM wave exclusively.

The general solution for the TM wave in free space when $\zeta_1 \leq \zeta \leq \zeta_2$

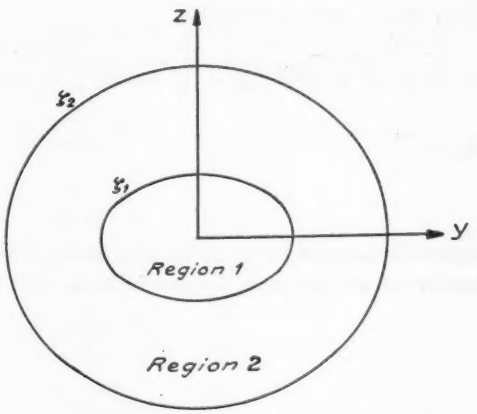


Fig. 2. The region 2 is defined by two confocal ellipses ζ_1 and ζ_2.

For this case (fig. 2) it is immediately possible to write the solutions in component form. It is only necessary to note that now both radial functions Je_l, Ne_l and Jo_l, No_l have to be taken into account since Ne_l and No_l can no longer become $-\infty$.

We obtain thus

$$
\begin{cases}
E_x = B_2 \begin{Bmatrix} Se_l\,(c, \cos v)\,[Je_l\,(c, \cosh \zeta) + p_e\, Ne_l\,(c, \cosh \zeta)] \\ So_l\,(c, \cos v)\,[Jo_l\,(c, \cosh \zeta) + p_0\, No_l\,(c, \cosh \zeta)] \end{Bmatrix} e^{j\omega t - \gamma x} \\
E_\zeta = + \dfrac{\gamma}{j\,\omega\,\varepsilon_0}\, H_v \\
E_v = - \dfrac{\gamma}{j\,\omega\,\varepsilon_0}\, H_\zeta \\
H_x = 0 \\
H_\zeta = + \dfrac{j\,\omega\,\varepsilon_0}{q_1\,(k_0^2 + \gamma^2)} \cdot \dfrac{\partial E_x}{\partial v} \\
H_v = - \dfrac{j\,\omega\,\varepsilon_0}{q_1\,(k_0^2 + \gamma^2)} \cdot \dfrac{\partial E_x}{\partial \zeta}
\end{cases}
\tag{11}
$$

where

p_e , p_0 are constants,

$c = q\,(k_0^2 + \gamma^2)^{1/2}$, $q_1^2 = q^2\,(\cosh^2 \zeta - \cos^2 v)$

$k_0 = \dfrac{\omega}{c_0}$

Since considerable use will be made in this work of Mathieu's functions, the symbols employed are given in Appendix 2, as well as some important formulas.

The elliptic electron beam in a confocal elliptic wave tube

Let us consider the case corresponding to fig. 3.

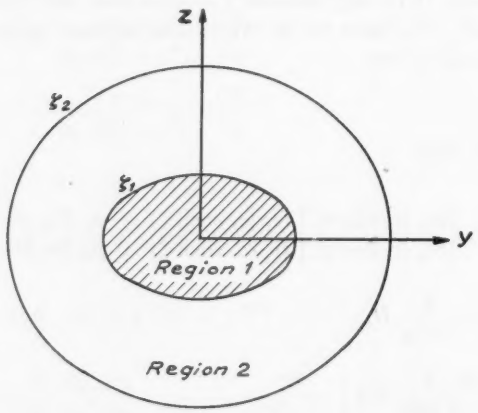


Fig. 3. Elliptic electron beam in a confocal elliptic wave tube.

On the basis of the foregoing it is immediately possible to write the solutions separately for region 1 and region 2. We are considering only the TM wave in this connection.

TM wave:

Region 1.

$$\begin{cases} E_x = B_1 \begin{Bmatrix} Se_l\,(c_1, \cos v)\; Je_l\,(c_1, \cosh \zeta) \\ So_l\,(c_1, \cos v)\; Jo_l\,(c_1, \cosh \zeta) \end{Bmatrix} e^{j\omega t - \gamma x} \\ E_\zeta = + \dfrac{\gamma}{j\,\omega\,\varepsilon_0}\, H_v \\ E_v = - \dfrac{\gamma}{j\,\omega\,\varepsilon_0}\, H_\zeta \\ H_x = 0 \\ H_\zeta = + \dfrac{j\,\omega\,\varepsilon_0}{q_1\,(k_0^2 + \gamma^2)} \cdot \dfrac{\partial E_x}{\partial v} \\ H_v = - \dfrac{j\,\omega\,\varepsilon_0}{q_1\,(k_0^2 + \gamma^2)} \cdot \dfrac{\partial E_x}{\partial \zeta} \end{cases} \tag{12}$$

where

$$c_1^2 = q^2\,\Lambda^2\,(k_0^2 + \gamma^2) \quad ; \quad \Lambda^2 = \left[1 - \frac{1}{2}\,\frac{Z_0}{z_0} \cdot \frac{c_0}{v_0} \cdot \frac{1}{(k_1 + j\,\gamma)^2}\right]$$

$$k_0^2 = \left(\frac{\omega}{c_0}\right)^2, \quad c_0 = \frac{1}{\sqrt{\mu_0\,\varepsilon_0}}$$

Region 2.

$$\begin{cases} E_x = B_2 \begin{Bmatrix} Se_l\,(c, \cos v)\,[Je_l\,(c, \cosh \zeta) + p_e\, Ne_l\,(c, \cosh \zeta)] \\ So_l\,(c, \cos v)\,[Jo_l\,(c, \cosh \zeta) + p_0\, No_l\,(c, \cosh \zeta)] \end{Bmatrix} e^{j\omega t - \gamma x} \\ E_\zeta = + \dfrac{\gamma}{j\,\omega\,\varepsilon_0}\, H_v \\ E_v = - \dfrac{\gamma}{j\,\omega\,\varepsilon_0}\, H_\zeta \\ H_x = 0 \\ H_\zeta = + \dfrac{j\,\omega\,\varepsilon_0}{q_1\,(k_0^2 + \gamma^2)} \cdot \dfrac{\partial E_x}{\partial v} \\ H_v = - \dfrac{j\,\omega\,\varepsilon_0}{q_1\,(k_0^2 + \gamma^2)} \cdot \dfrac{\partial E_x}{\partial \zeta} \end{cases} \tag{13}$$

where $c^2 = q^2\,(k_0^2 + \gamma^2)$

Boundary conditions

Generally, the tangential component of the electric field is continuous at the boundary surface of two media. This can be written $E_{t1} = E_{t2}$. With regard to the tangential component of the magnetic field at the boundary of two media there can generally be written $H_{t1} - H_{t2} = K$, where K is the linear density of any current flowing in an infinitesimally thin sheet of the boundary surface. Particularly in the case now under consideration, however, we impose the additional requirement that the impedance be the same on either side of the boundary surface. This implies that no power reflexions occur when a power flow in the direction of the normal upon the boundary surface is concerned. The impedance is the ratio of the tangential electric field and the tangential magnetic field. From this, it follows immediately in the matching that at the boundary surface the tangential components of the electric as well as the magnetic field have to be continuous. We may thus write

$$\left.\begin{aligned} E_{t1} &= E_{t2} \\ H_{t1} &= H_{t2} \end{aligned}\right\} \quad \therefore \quad \frac{E_{t1}}{H_{t1}} = \frac{E_{t2}}{H_{t2}} = Z \tag{14}$$

When applied to the TM wave in elliptic coordinates, the above-mentioned conditions can be written

$$\left\{\begin{aligned} E_x^1 &= E_x^2 \\ E_v^1 &= E_v^2 \\ H_v^1 &= H_v^2 \end{aligned}\right. \tag{15}$$

where the upper index 1 refers to region 1 and
the upper index 2 refers to region 2.

Matching at the surface ζ_1:

The requirement is that the following system of equations be satisfied at all values of v:

$$\left.\begin{aligned} E_x^1 &= E_x^2 \\ E_v^1 &= E_v^2 \\ H_v^1 &= H_v^2 \end{aligned}\right\} \quad \therefore \quad H_\zeta^1 = H_\zeta^2 \qquad \text{(See formulas 12 and 13)}$$

It should be noted that one of these equations could also be obtained from the general condition that the normal component of the flux density $\bar{B}$ has to be continuous between two boundary surfaces. This can be expressed as $B_{n1} = B_{n2}$, i. e., if $\mu_1 = \mu_2$, then $H_{\zeta 1} = H_{\zeta 2}$. Application of the above-mentioned conditions to the case in hand renders

$$
\begin{cases}
B_1 \begin{Bmatrix} Se_l\,(c_1,\,\cos v)\; Je_l\,(c_1,\,\cosh\zeta_1) \\ So_l\,(c_1,\,\cos v)\; Jo_l\,(c_1,\,\cosh\zeta_1) \end{Bmatrix} \\
= B_2 \begin{Bmatrix} Se_l\,(c,\,\cos v)\,[Je_l\,(c,\,\cosh\zeta_1) + p_e\, Ne_l\,(c,\,\cosh\zeta_1)] \\ So_l\,(c,\,\cos v)\,[Jo_l\,(c,\,\cosh\zeta_1) + p_0\, Ne_l\,(c,\,\cosh\zeta_1)] \end{Bmatrix} \\
B_1 \begin{Bmatrix} \dfrac{\partial}{\partial v}\, Se_l\,(c_1,\,\cos v)\; Je_l\,(c_1,\,\cosh\zeta_1) \\ \dfrac{\partial}{\partial v}\, So_l\,(c_1,\,\cos v)\; Jo_l\,(c_1,\,\cosh\zeta_1) \end{Bmatrix} \\
= B_2 \begin{Bmatrix} \dfrac{\partial}{\partial v}\, Se_l\,(c,\,\cos v)\,[Je_l\,(c,\,\cosh\zeta_1) + p_e\, Ne_l\,(c,\,\cosh\zeta_1)] \\ \dfrac{\partial}{\partial v}\, So_l\,(c,\,\cos v)\,[Jo_l\,(c,\,\cosh\zeta_1) + p_0\, No_l\,(c,\,\cosh\zeta_1)] \end{Bmatrix} \\
B_1 \begin{Bmatrix} Se_l\,(c_1,\,\cos v)\,\left[\dfrac{\partial}{\partial\zeta}\, Je_l\,(c_1,\,\cosh\zeta)\right]_{\zeta=\zeta_1} \\ So_l\,(c_1,\,\cos v)\,\left[\dfrac{\partial}{\partial\zeta}\, Jo_l\,(c_1,\,\cosh\zeta)\right]_{\zeta=\zeta_1} \end{Bmatrix} \\
= B_2 \begin{Bmatrix} Se_l\,(c,\,\cos v)\,\left[\dfrac{\partial}{\partial\zeta}\, Je_l\,(c,\,\cosh\zeta) + p_e\,\dfrac{\partial}{\partial\zeta}\, Ne_l\,(c,\,\cosh\zeta)\right]_{\zeta=\zeta_1} \\ So_l\,(c,\,\cos v)\,\left[\dfrac{\partial}{\partial\zeta}\, Jo_l\,(c,\,\cosh\zeta) + p_0\,\dfrac{\partial}{\partial\zeta}\, No_l\,(c,\,\cosh\zeta)\right]_{\zeta=\zeta_1} \end{Bmatrix}
\end{cases}
\tag{16}
$$

There is reason to note, furthermore, that in an elliptic wave tube the wave is split into odd and even waves. Consequently, they have to be treated separately. The boundary conditions, too, are therefore obtained separately for odd and even waves.

As boundary conditions on the surface ζ_1 we obtain for even waves:

$$
\begin{cases}
B_1\, Se_l\,(c_1,\,\cos v)\; Je_l\,(c_1,\,\cosh\zeta_1) \\
= B_2\, Se_l\,(c,\,\cos v)\,[Je_l\,(c,\,\cosh\zeta_1) + p_e\, Ne_l\,(c,\,\cosh\zeta_1)] \\
B_1\, Se_l\,(c_1,\,\cos v)\,\left[\dfrac{\partial}{\partial\zeta}\, Je_l\,(c_1,\,\cosh\zeta_1)\right]_{\zeta=\zeta_1} \\
= B_2\, Se_l\,(c,\,\cos v)\,\left[\dfrac{\partial}{\partial\zeta}\, Je_l\,(c,\,\cosh\zeta) + p_e\,\dfrac{\partial}{\partial\zeta}\, Ne_l\,(c,\,\cosh\zeta)\right]_{\zeta=\zeta_1}
\end{cases}
$$

This condition has to be satisfied at all values of v, i.e., also at $v = 0$. According to definition, then, $Se_l\,(c_1, 1) = 1$ and $Se_l\,(c, 1) = 1$. It should be noted that as a rule $\frac{d}{dv}\,Se_l\,(c, \cos v) \underset{v\to 0}{\longrightarrow} 0$ and $\frac{d}{dv}\,So_l\,(c, \cos v) \underset{v\to 0}{\longrightarrow} 1$. Thus, the second equation among the original boundary conditions does not add any new condition to those contained in the first two equations. Therefore, the following equations, independent of the angle v, are obtained as final boundary conditions for even waves on the surface:

$$\left\{\begin{aligned} & B_1\,Je_l\,(c_1, \cosh\zeta_1) \\ &= B_2\,[Je_l\,(c, \cosh\zeta_1) + p_e\,Ne_l\,(c, \cosh\zeta_1)] \\ & B_1\left[\frac{\partial}{\partial\zeta}\,Je_l\,(c_1, \cosh\zeta)\right]_{\zeta=\zeta_1} \\ &= B_2\left[\frac{\partial}{\partial\zeta}\,Je_l\,(c, \cosh\zeta) + p_e\,\frac{\partial}{\partial\zeta}\,Ne_l\,(c, \cosh\zeta)\right]_{\zeta=\zeta_1} \end{aligned}\right. \tag{17}$$

where B_1, B_2 and p_e are constants.

It is obvious that analogous equations are obtained for odd waves. We shall state them for the sake of completeness:

$$\left\{\begin{aligned} & B_1\,Jo_l\,(c_1, \cosh\zeta_1) \\ &= B_2\,[Jo_l\,(c, \cosh\zeta_1) + p_0\,No_l\,(c, \cosh\zeta_1)] \\ & B_1\left[\frac{\partial}{\partial\zeta}\,Jo_l\,(c_1, \cosh\zeta)\right]_{\zeta=\zeta_1} \\ &= B_2\left[\frac{\partial}{\partial\zeta}\,Jo_l\,(c, \cosh\zeta) + p_0\,\frac{\partial}{\partial\zeta}\,No_l\,(c, \cosh\zeta)\right]_{\zeta=\zeta_1} \end{aligned}\right. \tag{18}$$

Matching at the surface ζ_2:

The conductivity of the tube at the surface ζ_2 is assumed to be infinite ($\sigma = \infty$). Then, the tangential component of the electric field has to vanish. In the case in hand this condition implies that

$$\left.\begin{aligned} E_x &= 0 \\ E_v &= 0 \end{aligned}\right\} \text{ on the surface } \zeta = \zeta_2.$$

It should be noted that the latter equation immediately follows from the condition $E_x = 0$. This is seen from equations 13. One of these two equations is thus sufficient.

In this manner the following boundary condition is obtained for even waves:

$$B_2\, Se_l\,(c, \cos v)\,[Je_l\,(c, \cosh \zeta_2) + p_e\, Ne_l\,(c, \cosh \zeta_2)] = 0$$

As the boundary condition independent of the angle v we have thus

$$Je_l\,(c, \cosh \zeta_2) + p_e\, Ne_l\,(c, \cosh \zeta_2) = 0 \tag{19}$$

For odd waves, the analogous boundary condition on the surface ζ_2 is obtained:

$$Jo_l\,(c, \cosh \zeta_2) + p_0\, No_l\,(c, \cosh \zeta_2) = 0 \tag{20}$$

Let us now assemble the boundary conditions found above for even and odd waves.

Boundary conditions for even waves:

$$\begin{cases} B_1\, Je_l\,(c_1, \cosh \zeta_1) \\ = B_2\,[Je_l\,(c, \cosh \zeta_1) + p_e\, Ne_l\,(c, \cosh \zeta_1)] \\ B_1 \left[\dfrac{\partial}{\partial \zeta}\, Je_l\,(c_1, \cosh \zeta)\right]_{\zeta = \zeta_1} \\ = B_2 \left[\dfrac{\partial}{\partial \zeta}\, Je_l\,(c, \cosh \zeta) + p_e\, \dfrac{\partial}{\partial \zeta}\, Ne_l\,(c, \cosh \zeta)\right]_{\zeta = \zeta_1} \\ Je_l\,(c, \cosh \zeta_2) + p_e\, Ne_l\,(c, \cosh \zeta_2) = 0 \end{cases} \tag{21}$$

Boundary conditions for odd waves:

$$\begin{cases} B_1\, Jo_l\,(c_1, \cosh \zeta_1) \\ = B_2\,[Jo_l\,(c, \cosh \zeta_1) + p_0\, No_l\,(c, \cosh \zeta_1)] \\ B_1 \left[\dfrac{\partial}{\partial \zeta}\, Jo_l\,(c_1, \cosh \zeta)\right]_{\zeta = \zeta_1} \\ = B_2 \left[\dfrac{\partial}{\partial \zeta}\, Jo_l\,(c, \cosh \zeta) + p_0\, \dfrac{\partial}{\partial \zeta}\, No_l\,(c, \cosh \zeta)\right]_{\zeta = \zeta_1} \\ Jo_l\,(c, \cosh \zeta_2) + p_0\, No_l\,(c, \cosh \zeta_2) = 0 \end{cases} \tag{22}$$

Determination of the propagation constant

We shall here concern ourselves with even waves. It is obvious that analogous solutions are obtained for odd waves. From the boundary conditions derived for even waves, the following equation is obtained:

$$\left\{ \frac{\left[\frac{\partial}{\partial\zeta} Je_l\,(c_1, \cosh\zeta)\right]_{\zeta=\zeta_1}}{Je_l\,(c_1, \cosh\zeta_1)} = \frac{\left[\frac{\partial}{\partial\zeta} Je_l\,(c, \cosh\zeta) - \frac{Je_l\,(c, \cosh\zeta_2)}{Ne_l\,(c, \cosh\zeta_2)} \frac{\partial}{\partial\zeta} Ne_l\,(c, \cosh\zeta)\right]_{\zeta=\zeta_1}}{Je_l\,(c, \cosh\zeta_1) - \frac{Je_l\,(c, \cosh\zeta_2)}{Ne_l\,(c, \cosh\zeta_2)} Ne_l\,(c, \cosh\zeta_1)} \right. \tag{23}$$

It can be concluded from a study of this equation, which is of the form $f_1(c_1) = f(c)$, that it has no real solutions in the given form. For this purpose it is necessary to assume that c_1 and c are close to each other. It will be found that such an assumption is justified in practice. It should further be remembered that the value of the Wronskian of the radial function is 1. The real solutions of the equation are found easily, in principle at least, by assuming that c and Λ have the form

$$\begin{aligned} c &= j\,q\,h_0 \\ \Lambda &= j\,D \end{aligned} \quad \therefore \quad c_1 = \Lambda c = -\,q\,Dh_0 = -\,qH \quad \text{where } H = Dh_0 \tag{24}$$

On the other hand this implies that

$$c = q\,(k_0^2 + \gamma^2)^{1/2} = j\,q\,(-\,\gamma^2 - k_0^2)^{1/2} = j\,q\,h_0$$

$$\Lambda = \left[1 - \frac{a^2}{(k_1 + j\,\gamma)^2}\right]^{1/2} = j\left[\frac{a^2}{(k_1 + j\,\gamma)^2} - 1\right]^{1/2} = j\,D$$

where

$$h_0 = (-\,\gamma^2 - k_0^2)^{1/2} = (\beta^2 - k_0^2)^{1/2}, \quad \text{where} \quad \alpha = 0$$

$$D = \left[\left(\frac{a}{(k_1 + j\,\gamma)}\right)^2 - 1\right]^{1/2}; \quad a^2 = \frac{1}{2} \cdot \frac{Z_0}{z_0} \cdot \frac{c_0}{v_0}$$

$$H = Dh_0$$

The equation under consideration now takes the form

$$\left\{\begin{array}{l} \dfrac{\left[\dfrac{\partial}{\partial\zeta} Je_l\,(-qH,\ \cosh\zeta)\right]_{\zeta=\zeta_1}}{Je_l\,(-qH,\ \cosh\zeta_1)} \\ = \dfrac{\left[\dfrac{\partial}{\partial\zeta} Ie_l\,(qh_0,\ \cosh\zeta) - \dfrac{Ie_l\,(qh_0,\ \cosh\zeta_2)}{Me_l\,(qh_0,\ \cosh\zeta_2)} \dfrac{\partial}{\partial\zeta} Me_l\,(qh_0,\ \cosh\zeta)\right]_{\zeta=\zeta_1}}{Ie_l\,(qh_0,\ \cosh\zeta_1) - \dfrac{Ie_l\,(qh_0,\ \cosh\zeta_2)}{Me_l\,(qh_0,\ \cosh\zeta_2)} Me_l\,(qh_0,\ \cosh\zeta_1)} \end{array}\right. \tag{25}$$

Here, we have introduced the notations

$$\left\{\begin{array}{l} Je_l\,(jc,\ z) = Ie_l\ (c,\ z) = \sqrt{\dfrac{\pi}{2}} \sum\limits_{n=0,1}^{\infty}{}' j^l\, D_n^l\, I_n\,(cz) \\ Ne_l\,(jc,\ z) = Me_l\ (c,\ z) = \sqrt{\dfrac{2}{\pi}} \sum\limits_{n=0,1}^{\infty}{}' j^l\, D_n^l \left[(-1)^{n+1} K_n\,(cz) + j\,\dfrac{\pi}{2}\, I_n\,(cz)\right] \end{array}\right. \tag{26}$$

It should be noted that

$$\begin{aligned} N_n\,(jz) &= -\frac{2}{\pi}\, j^{-n}\, K_n\,(z) + j^{n+1}\, I_n\,(z) \\ &= (-1)^{n+1} \cdot \frac{2}{\pi} \cdot j^n\, K_n\,(z) + j^{n+1}\, I_n\,(z) \end{aligned}$$

Hence it follows that $Ne_l\,(jc, z)$ is split into two parts

$$Ne_l\,(jc, z) = Me_l\,(c, z) = Ke_l\,(c, z) + j\, Ie_l\,(c, z) \tag{27}$$

where

$$Ke_l\,(c, z) = -\sqrt{\frac{2}{\pi}} \sum_{n=0,1}^{\infty}{}' j^l\, D_n^l\, (-1)^n\, K_n\,(c\,z)$$

Study of the obtained boundary condition equation reveals that its left member is real, whereas its right member is a complex magnitude in general, owing to the fact that $Me_l\,(c, z)$ is complex at all values of l. Obviously, then, two equations result from this one equation, one of them relating to the real parts, the other one to the coefficients of the imaginary parts. However, it is to be remembered that this equation was the result of a consideration of physical quantities. Therefore, one should not forget that in such a case only the real part of the complex quantity is considered. Let us now

examine the obtained expressions for boundary conditions of even waves. With regard to the constant p_e we impose the limitation that it has to be real. From a physical point of view this is expedient. It follows then immediately that of the expression of $Ne_l\ (jc,\ z)$ only the term $Ke_l\ (c,\ z)$ has to be taken into account. This is because the value of p_e is obtained from the boundary condition which states that E_x must vanish on the surface of the conducting tube. Consequently, $p_e = -\dfrac{Ie_l\ (c,\ \cosh\ \zeta_2)}{Ne_l\ (jc,\ \cosh\ \zeta_2)}$.

Generally, $\dfrac{Ie_l\ (c,\ z)}{Ke_l\ (c,\ z)} = -\dfrac{\pi}{2} \cdot \dfrac{\sum\limits_{n=0,1}^{\infty}{}' D_n^l\ I_n\ (c\ z)}{\sum\limits_{n=0,1}^{\infty}{}' (-1)^n\ D_n^l\ K_n\ (c\ z)} =$ real at all values of l.

On the other hand there would be $p_e = -j$ if the imaginary part of Me_l would be taken into account. From a physical point of view, thus, the limitation is justified which we are going to make here, namely, that of the $Ne_l\ (jc,\ z)$ term only the part $Ke_l\ (c,\ z)$ is taken into account. Consequently, we write

$$\text{Physical quantity } [Ne_l\ (jc,\ z)] = Re\ [Ne_l\ (jc,\ z)] = Ke_l\ (c,\ z)$$

Our boundary condition equation can now be written in the following form, considering the said limitations:

$$\left\{ \frac{\left[\dfrac{\partial}{\partial\zeta}\ Je_l\ (-\ qH,\ \cosh\ \zeta)\right]_{\zeta=\zeta_1}}{Je_l\ (-\ qH,\ \cosh\ \zeta_1)} \right. \tag{28}$$

$$\left. = \frac{\left[\dfrac{\partial}{\partial\zeta}\ Ie_l\ (qh_0,\ \cosh\ \zeta) - \dfrac{Ie_l\ (qh_0,\ \cosh\ \zeta_2)}{Ke_l\ (qh_0,\ \cosh\ \zeta_2)}\ \dfrac{\partial}{\partial\zeta}\ Ke_l\ (qh_0,\ \cosh\ \zeta)\right]_{\zeta=\zeta_1}}{Ie_l\ (qh_0,\ \cosh\ \zeta_1) - \dfrac{Ie_l\ (qh_0,\ \cosh\ \zeta_2)}{Ke_l\ (qh_0,\ \cosh\ \zeta_2)}\ Ke_l\ (qh_0,\ \cosh\ \zeta_1)} \right.$$

If we now examine the equation obtained in this way, we find that its left member is oscillating and its right member nonoscillating. It follows immediately that real solutions exist. Let us denote the left member with $F\ (qH)$ and the right member with $f\ (qh_0)$. The equation can thus be written in the form

$$F\,(qH) = f\,(qh_0)$$

The equation under consideration can be solved graphically. In principle, $F\,(qH)$ is of the form shown in fig. 4.

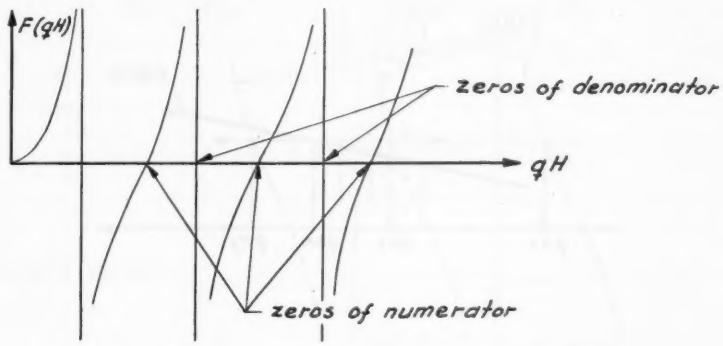


Fig. 4. *F (qH)* as a function of qH. Principle diagram.

The right member, again, is in principle a monotonously increasing function. It has the form

$$f\,(qh_0) = \frac{\left[\frac{\partial}{\partial\zeta}\,Ke_l\,(qh_0,\,\cosh\,\zeta) - \frac{Ke_l\,(qh_0,\,\cosh\,\zeta_2)}{Ie_l\,(qh_0,\,\cosh\,\zeta_2)}\,\frac{\partial}{\partial\zeta}\,Ie_l\,(qh_0,\,\cosh\,\zeta)\right]_{\zeta=\zeta_1}}{Ke_l\,(qh_0,\,\cosh\,\zeta_1) - \frac{Ke_l\,(qh_0,\,\cosh\,\zeta_2)}{Ie_l\,(qh_0,\,\cosh\,\zeta_2)}\,Ie_l\,(qh_0,\,\cosh\,\zeta_1)}$$

This function is shown in fig. 5.

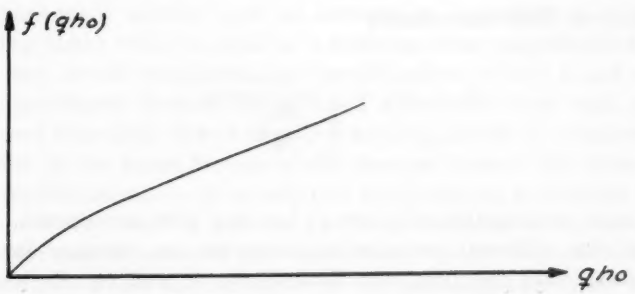


Fig. 5. $f\,(qh_0)$ as a function of qh_0. Principle diagram.

Obviously thus for a fixed value of qh_0, $f\ (qh_0)$ takes on a definite value. On the other hand an entire infinite series of qH values in the function $F\ (qH)$ correspond to this value. Let us denote them with qH_n. Graphically this can be illustrated by figure 6.

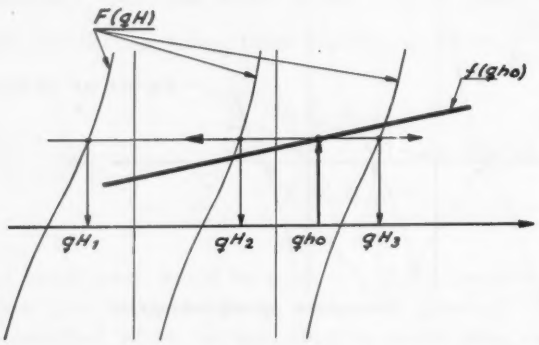


Fig. 6. To a given value of qh_0 corresponds an infinite series of values qH which are denoted with qH_n. Principle diagram.

Obviously then

$$H_n = h_0\ D_n \tag{29}$$

Taking into account the employed notations, we obtain

$$H_n = \pm\ h_0 \sqrt{\left(\frac{a}{k_1 + j\ \gamma_n}\right)^2 - 1} \tag{30}$$

From this, we find immediately

$$k_1 + j\ \gamma_n = \frac{\pm\ a}{\sqrt{\left(\frac{H_n}{h_0}\right)^2 + 1}} \tag{31}$$

The axial propagation constant γ_n has now been determined. It should be noted that different propagation constants are obtained for odd and even waves, since the functions Je_l and Jo_l, Ie_l and Io_l, Ke_l and Ko_l are different from each other. Actually, this signifies that the wave is split into

odd and even waves which differ from each other in regard to their phase factor. The damping factor was assumed to be zero ($\alpha = 0$) in both cases.

For even waves

$$k_1 + j\,\gamma^{(e)}_{\pm n} = \frac{\pm a}{\sqrt{\left(\frac{H^{(e)}_n}{h_0}\right)^2 + 1}} = \triangle\,\gamma^{(e)}_{\pm n} \tag{32}$$

Thus

$$-\gamma^{(e)}_{\pm n} = \pm j\,\frac{a}{\sqrt{\left(\frac{H^{(e)}_n}{h_0}\right)^2 + 1}} - j\,k_1$$

For odd waves

$$k_1 + j\,\gamma^{(o)}_{\pm n} = \frac{\pm a}{\sqrt{\left(\frac{H^{(o)}_n}{h_0}\right)^2 + 1}} = \triangle\,\gamma^{(e)}_{\pm n} \tag{33}$$

Thus

$$-\gamma^{(o)}_{\pm n} = \pm j\,\frac{a}{\sqrt{\left(\frac{H^{(o)}_n}{h_0}\right)^2 + 1}} - j\,k_1$$

It is seen that the expressions are completely analogous to those in the case of circular cross section.

There is reason to observe that the calculations performed above are based on the assumption that the damping factor α equals zero. Now the question rears itself whether such an assumption is justified since it is just the damping factor which is used as a criterion when amplification occurs. However, one should remember that amplification is only found when at least two modulated electron beams exist, interacting with each other. It will be shown later that then a negative damping factor is obtained which is a function of the phase factors of the electron beams. We observe thus that the simplification $\alpha = 0$ assumed in the foregoing is justified since in every case α is very small in practice and since furthermore this assumption is essential for simplification of the calculations. Rather tedious work would be involved if one would have to deal with the functions which have occurred in the foregoing when the variable is complex.

The beam waves

Let us assume that there is one modulated electron beam, and let us examine the TM wave. Let the modulator grid be located at $x = 0$. We assume further that $|v|\ e^{j\omega t}$ is constant at a given distance x from the modulator grid over the entire elliptic cross section. There is no A. C. current in the plane $x = 0$, and therefore $J_x\ (x = 0) = 0$.

The axial motion equation is

$$m \frac{d^2x}{dt^2} = - e\ E_x = m \frac{d}{dt} (v_0 + v)$$

From this, the following expression was derived for v:

$$v = + j \frac{e}{\omega m} E_x \cdot \frac{k_1}{k_1 + j \gamma}$$

In the electron beam, E_x has the form

$$E_x = B_1 \begin{Bmatrix} Se_l\ (c_1,\ \cos v)\ Je_l\ (c_1,\ \cosh \zeta) \\ So_l\ (c_1,\ \cos v)\ Jo_l\ (c_1,\ \cosh \zeta) \end{Bmatrix} e^{j\omega t - \gamma x} \tag{34}$$

where

$$c_1 = \Lambda\ q\ (k_0^2 + \gamma^2)^{1/2} = \Lambda \cdot c$$

It was found that there exists an entire series of values for the propagation constant γ. These values were denoted by γ_n. For this reason the total value of E_x is

$$E_x\ (x,\ c_1,\ v,\ \zeta) = \sum_{n=-\infty}^{+\infty} B_{1_n} \begin{Bmatrix} Se_l\ (c_1,\ \cos v)\ Je_l\ (c_1,\ \cosh \zeta) \\ So_l\ (c_1,\ \cos v)\ Jo_l\ (c_1,\ \cosh \zeta) \end{Bmatrix} e^{j\omega t - \gamma_n x} \tag{35}$$

Correspondingly,

$$v = + j \frac{e}{\omega m} \sum_{n=-\infty}^{+\infty} \frac{k_1}{k_1 + j \gamma_n} B_{1_n} \begin{Bmatrix} Se_l\ (c_1,\ \cos v)\ Je_l\ (c_1,\ \cosh \zeta) \\ So_l\ (c_1,\ \cos v)\ Je_l\ (c_1,\ \cosh \zeta) \end{Bmatrix} e^{j\omega t - \gamma_n x} \tag{36}$$

and

$$J_x = - j \frac{1}{2 z_0} \sum_{n=-\infty}^{+\infty} \frac{k_1}{(k_1 + j \gamma_n)^2} B_{1_n} \begin{Bmatrix} Se_l\ (c_1, \cos v)\ Je_l\ (c_1, \cosh \zeta) \\ So_l\ (c_1, \cos v)\ Jo_l\ (c_1, \cosh \zeta) \end{Bmatrix} e^{j\omega t - \gamma_n x} \tag{37}$$

It was found that $J_x\,(x = 0) = 0$. Consequently,

$$\sum_{n=-\infty}^{+\infty} \frac{k_1}{(k_1 + j\,\gamma_n)^2}\, B_{\underset{n}{1}} = 0 \tag{38}$$

Let us then examine the derived boundary condition equation. It consists of an oscillating part and a non-oscillating part. It should be noted with regard to the former that its value does not change even if the sign of the argument changes. This is a consequence of the well-known equation

$$J_{-n}\,(z) = J_n\,(-z) = (-1)^n\, J_n\,(z)$$

When applied to radial functions of the first kind, this implies that

$$Je_l\,(-z) = +\,Je_l\,(z) \quad ; \quad \text{if } l = \text{even}$$

$$Je_l\,(-z) = -\,Je_l\,(z) \quad ; \quad \text{if } l = \text{odd}$$

Hence it follows that

$$H_{+n} = H_{-n} \;\therefore\; \Delta\,\gamma_{+n} = -\,\Delta\,\gamma_{-n} \quad \text{or} \quad k_1 + j\,\gamma_{+n} = -\,(k_1 + j\,\gamma_{-n})$$

This, again, implies that from equation (38) the following conclusion can be drawn:

$$B_{\underset{n}{1}} = -\,B_{\underset{-n}{1}} \tag{39}$$

Furthermore, we may write

$$\begin{cases} -\,\gamma_{+n} = +\,j\,\dfrac{\alpha}{\sqrt{\left(\dfrac{H_n}{h_0}\right)^2 + 1}} - j\,k_1 = +\,j\,\Delta\,\gamma_{+n} - j\,k_1 \\[2ex] -\,\gamma_{-n} = -\,j\,\dfrac{\alpha}{\sqrt{\left(\dfrac{H_{-n}}{h_0}\right)^2 + 1}} - j\,k_1 = -\,j\,\Delta\,\gamma_{+n} - j\,k_1 \end{cases} \tag{40}$$

It is easy to notice that E_x, v and J_x can be written in a slightly different form if one takes into account that

$$e^{+j\triangle\gamma_n x} + e^{-j\triangle\gamma_n x} = 2\cos(\triangle\gamma_n x)$$

$$e^{+j\triangle\gamma_n x} - e^{-j\triangle\gamma_n x} = 2j\sin(\triangle\gamma_n x)$$

We obtain thus

$$\begin{cases} E_x = +2j\,e^{-jk_1 x}\sum_{n=1}^{\infty} B_{1_n}\sin(\triangle\gamma_n x)\begin{Bmatrix} Se_l(c_1,\cos v)\,Je_l(c_1,\cosh\zeta) \\ So_l(c_1,\cos v)\,Jo_l(c_1,\cosh\zeta)\end{Bmatrix} e^{j\omega t} \\ v = +2j\frac{e}{\omega m}\cdot e^{-jk_1 x}\sum_{n=1}^{\infty} B_{1_n}\frac{k_1}{\triangle\gamma_n}\cos(\triangle\gamma_n x)\begin{Bmatrix} Se_l(c_1,\cos v)\,Je_l(c_1,\cosh\zeta) \\ So_l(c_1,\cos v)\,Jo_l(c_1,\cosh\zeta)\end{Bmatrix} e^{j\omega t} \\ J_x = \frac{1}{z_0}\cdot e^{-jk_1 x}\sum_{n=1}^{\infty} B_{1_n}\frac{k_1}{(\triangle\gamma_n)^2}\sin(\triangle\gamma_n x)\begin{Bmatrix} Se_l(c_1,\cos v)\,Je_l(c_1,\cosh\zeta) \\ So_l(c_1,\cos v)\,Jo_l(c_1,\cosh\zeta)\end{Bmatrix} e^{j\omega t} \end{cases} \tag{41}$$

The value of the total current will be

$$I_{xT} = \int_0^{\zeta_1}\int_0^{2\pi} J_x\left\{q^2\cosh^2\zeta\left[1-\left(\frac{\cos v}{\cosh\zeta}\right)^2\right]dv\right\}d\zeta \tag{42}$$

It should be noted with regard to the surface integral that

$da_1 = \sqrt{g_{22}\,g_{33} - g_{23}^2}\,du^2\,du^3$ [Stratton (32), p. 42]

Hence

$$da_1 = \sqrt{h_2^2\,h_3^2}\,d\zeta\,dv = h_2\,h_3\,d\zeta\,dv = q_1^2\,d\zeta\,dv = q^2(\cosh^2\zeta - \cos^2 v)\,d\zeta\,dv$$

$$= q_1^2\cosh^2\zeta\left[1-\left(\frac{\cos v}{\cosh\zeta}\right)^2\right]d\zeta\,dv$$

Obviously, further,

$$a_{1\,\mathrm{ell}} = \int_0^{\zeta_1}\int_0^{2\pi}\left\{q^2\cosh^2\zeta\left[1-\left(\frac{\cos v}{\cosh\zeta}\right)^2\right]dv\right\}d\zeta = \text{area of the ellipse}$$

Determination of coefficient B_{1_n}:

Let us consider the case shown in fig. 7.

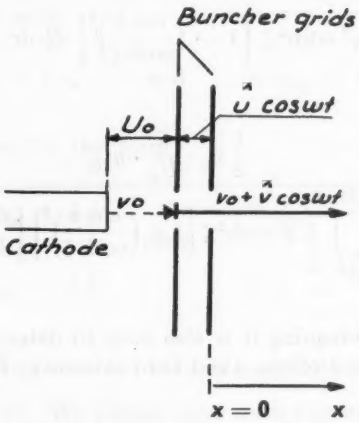


Fig. 7. Schematic diagram showing how velocity modulation is accomplished.

With the notations introduced in the foregoing, we obtain in the point $x = 0$ when the electron emerges from the buncher grids at time $t = 0$:

$$\begin{cases} \frac{1}{2}\, m\, v_0^2 = e\, U_0 \\ \frac{1}{2}\, m\, (v_0 + \hat{v})^2 = e\, (U_0 + \hat{u}) \end{cases} \tag{43}$$

This yields for the first approximation of $\hat{v}$:

$$\hat{v} = \frac{1}{2}\, v_0\, \frac{\hat{u}}{U_0} \qquad \text{at } x = 0$$

Consequently, we may write

$$\acute{v} = \frac{1}{2}\, v_0\, \frac{\hat{u}}{U_0} = +\, 2\, j\, \frac{e}{\omega\, m}\, B_{1_n} \cdot \frac{k_1}{\triangle\, \gamma_n} \begin{Bmatrix} Se_l\, (c_1,\, \cos\, v)\, Je_l\, (c_1,\, \cosh\, \zeta) \\ So_l\, (c_1,\, \cos\, v)\, Jo_l\, (c_1,\, \cosh\, \zeta) \end{Bmatrix} \tag{44}$$

In order to determine B_{1_n}, we eliminate ζ and v. This is accomplished by integrating the obtained expression over the elliptic cross section.

$$\frac{1}{2}\, v_0 \cdot \frac{\hat{u}}{U_0} \cdot a_{1\,\mathrm{ell}} = +\,2\,j\,\frac{e}{\omega\, m}\, B_{1\atop n} \cdot \frac{k_1}{\triangle\,\gamma_n} \int_0^{\zeta_1} \int_0^{2\pi} \begin{Bmatrix} Se_l\,(\,)\,Je_l\,(\,) \\ So_l\,(\,)\,Jo_l\,(\,) \end{Bmatrix} \cdot$$

$$\cdot\, q^2 \cosh^2 \zeta \left[1 - \left(\frac{\cos v}{\cosh \zeta}\right)^2\right] d\zeta\, dv$$

$$\therefore B_{1\atop n} = \frac{\frac{1}{2}\, v_0 \cdot \frac{\hat{u}}{U_0} \cdot a_{1\,\mathrm{ell}}}{+\,j\,\frac{2\,e}{\omega\, m} \cdot \frac{k_1}{\triangle\,\gamma_n} \int_0^{\zeta_1} \int_0^{2\pi} q^2 \cosh^2 \zeta \left[1 - \left(\frac{\cos v}{\cosh \zeta}\right)^2\right] \begin{Bmatrix} Se_l\,(\,)\,Je_l\,(\,) \\ So_l\,(\,)\,Jo_l\,(\,) \end{Bmatrix} d\zeta\, dv} \tag{45}$$

On the basis of the foregoing it is also easy to determine the peak values of the total current and of the axial field intensity, $\hat{I}$ and $\hat{E}_{xT}$:

$$\begin{cases} \hat{I}_{xT} = -\,j\,\dfrac{1}{2\,z_0} \cdot e^{-j k_1 x}\, \dfrac{\frac{1}{2}\, v_0\, \frac{\hat{u}}{U_0} \cdot a_{1\,\mathrm{ell}}}{\frac{e}{\omega\, m}} \sum_{n=1}^{\infty} \dfrac{\sin\,(\triangle\,\gamma_n\, x)}{\triangle\,\gamma_n} \\ \hat{E}_{xT} = +\,e^{-j k_1 x}\, \dfrac{\frac{1}{2}\, v_0\, \frac{\hat{u}}{U_0} \cdot a_{1\,\mathrm{ell}}}{\frac{e}{\omega\, m} \cdot k_1} \sum_{n=1}^{\infty} \triangle\,\gamma_n \sin\,(\triangle\,\gamma_n\, x) \end{cases} \tag{46}$$

Examining the expressions derived above, we find that no amplification is observable where one single modulated electron beam is concerned. This is quite natural indeed. However, the expressions just presented will be of use in the study of the case where two modulated electron beams are completely mixed. It should be noted that odd and even waves have to be treated separately.

Two completely mixed electron beams

It is easy to expand the theory presented in the foregoing so that it applies to two electron beams with the velocities v_1 and v_2. The quantities referring to beam No. 1 are denoted with subindex 1 and those pertaining to beam No. 2 with the index 2. Extension of formula (30) derived in the foregoing now yields

$$D_n = \frac{H_n}{h_0} = \sqrt{\frac{a_1^2}{(k_{01} + j\,\gamma_n)^2} + \frac{a_2^2}{(k_{02} + j\,\gamma_n)^2} - 1} \tag{47}$$

Furthermore, let us write

$$\triangle \gamma_{1n} = k_{01} + j\,\gamma_n \qquad \text{and} \qquad \triangle \gamma_{2n} = k_{02} + j\,\gamma_n$$

This can be expressed in the form

$$\begin{cases} \triangle \gamma_{1n} = \triangle \gamma_{0n} + \triangle k \\ \triangle \gamma_{2n} = \triangle \gamma_{0n} - \triangle k \end{cases} \qquad \text{where} \qquad \begin{aligned} \triangle \gamma_{0n} &= \frac{k_{01} + k_{02}}{2} + j\,\gamma_n \\ \triangle k &= \frac{k_{01} - k_{02}}{2} \end{aligned}$$

Let us further assume for the sake of simplicity that in the interaction region $a_1^2 = a_2^2 = \frac{1}{2}\,a^2$. We obtain then from equation (47):

$$\frac{1}{(\triangle \gamma_{0n} + \triangle k)^2} + \frac{1}{(\triangle \gamma_{0n} - \triangle k)^2} = 2\,\frac{\left(\frac{H_n}{h_0}\right)^2 + 1}{a^2} = \frac{1}{\delta_n^2} \tag{48}$$

$$\therefore\ \delta_n^2 = \frac{a^2}{2\left[\left(\frac{H_n}{h_0}\right)^2 + 1\right]}$$

We obtain further from equation (48)

$$\frac{1}{(y+1)^2} + \frac{1}{(y-1)^2} = w_n^2$$

where

$$\frac{\triangle \gamma_{0n}}{\triangle k} = y \quad \text{and} \quad \frac{\triangle k}{\delta_n} = w_n$$

As a series expansion, this is

$$y^4 - 2\,y^2\left(1 + \frac{1}{w_n^2}\right) + 1 - \frac{2}{w_n^2} = 0$$

$$\therefore\ y^2 = \frac{1}{w_n^2}\left(1 + w_n^2 \pm \sqrt{1 + 4\,w_n^2}\right) = \left(\frac{\triangle \gamma_{0n}}{\triangle k}\right)^2$$

$$\therefore\ (\triangle \gamma_{0n})^2 = \delta_n^2\,(1 + w_n^2 \pm \sqrt{1 + 4\,w_n^2}) \tag{49}$$

Four solutions are thus obtained for $\triangle \gamma_{0n}$. They are:

$$\begin{cases} \triangle \gamma_{0n1} = +\,\delta_n \sqrt{1 + w_n^2 + \sqrt{1 + 4\,w_n^2}} \\ \triangle \gamma_{0n2} = -\,\delta_n \sqrt{1 + w_n^2 + \sqrt{1 + 4\,w_n^2}} \\ \triangle \gamma_{0n3} = +\,\delta_n \sqrt{1 + w_n^2 - \sqrt{1 + 4\,w_n^2}} \\ \triangle \gamma_{0n4} = -\,\delta_n \sqrt{1 + w_n^2 - \sqrt{1 + 4\,w_n^2}} \end{cases} \tag{50}$$

According to the notations employed, there is

$$\triangle \gamma_{0n} = \frac{k_{01} + k_{02}}{2} + j\,\gamma_n = \frac{k_{01} + k_{02}}{2} + j\,\alpha_n - \beta_n, \text{ where } \gamma_n = \alpha_n + j\,\beta_n$$

Furthermore, $e^{-\gamma_n \cdot x} = e^{-\alpha_n \cdot x} \cdot e^{-j\beta_n \cdot x}$. Amplification is thus observable whenever $-\alpha_n$ = a positive real number. The amplification criterion may be presented in the form

$$\alpha_n < 0 \tag{51}$$

This, again, requires that

$$-\alpha_n = +\,j\,\triangle \gamma_{0n} - j\left(\frac{k_{01} + k_{02}}{2} - \beta_n\right) > 0 \tag{52}$$

This is possible only on condition that simultaneously

$$\begin{cases} \triangle \gamma_{0n} = -\,jK \\ \beta_n = \dfrac{k_{01} + k_{02}}{2} \end{cases} \quad \text{where } K = \text{positive real number} \tag{53}$$

It is obvious that only $\triangle \gamma_{0n4}$ can comply with these conditions. The condition for this to be true is

$$1 + 4\,w_n^2 > (1 + w_n^2)^2 = 1 + 2\,w_n^2 + w_n^4;$$

$$\therefore \; |w_n| < \sqrt{2} \tag{54}$$

The interaction is thus determined by the expression

$$|w_n| = \left| \frac{\triangle k}{\delta_n} \right| \leq \sqrt{2} \quad \text{or} \quad \left| \frac{k_{01} - k_{02}}{2} \right| \leq |\delta_n| \cdot \sqrt{2} \tag{55}$$

The amplification will obtain its highest value simultaneously with $|\triangle \gamma_{0n4}|$. Therefore, the problem goes back to that of determining the maximum of the function $f(w) = +\sqrt{1 + 4\,w^2} - (1 + w^2)$. This function has a maximum for $w^2 = \frac{3}{4}$ or $w = \frac{1}{2}\sqrt{3}$. Consequently, we have

$$|\triangle \gamma_{0n4}|_{\max} = |\delta_n| \cdot \left| \sqrt{1 + \frac{3}{4}} - 2 \right| = \frac{1}{2} \cdot |\delta_n|$$

Fig. 8 illustrates the behaviour of $\triangle \gamma_{0n4}/\delta_n$ as a function of w_n^2. According to the notations employed,

$$j\,\gamma_n = \triangle \gamma_{1n} - k_{01} = \triangle \gamma_{2n} - k_{02} = \triangle \gamma_{0n} - \frac{k_{01} + k_{02}}{2}$$

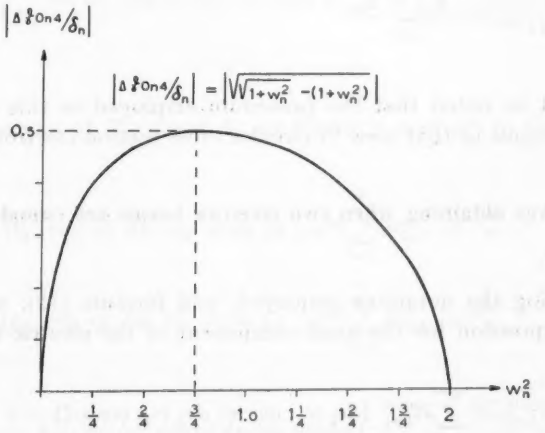


Fig. 8. $|\triangle \gamma_{0n4}/\delta_n| = f(w_n^2)$

Considering the values found for the propagation constant, we arrive at the following formulas in the region of interaction:

$$\left\{\begin{array}{l}
j\,\gamma_{n1} = +\,\delta_n \sqrt{1 + w_n^2 + \sqrt{1 + 4\,w_n^2}} - \dfrac{k_{01} + k_{02}}{2} = \\
\qquad = j\,\alpha_{n1} - \beta_{n1} \;\therefore\; \alpha_{n1} = 0 \\
j\,\gamma_{n2} = -\,\delta_n \sqrt{1 + w_n^2 + \sqrt{1 + 4\,w_n^2}} - \dfrac{k_{01} + k_{02}}{2} = \\
\qquad = j\,\alpha_{n2} - \beta_{n2} \;\therefore\; \alpha_{n2} = 0 \\
j\,\gamma_{n3} = +\,\delta_n \sqrt{1 + w_n^2 - \sqrt{1 + 4\,w_n^2}} - \dfrac{k_{01} + k_{02}}{2} = \\
\qquad = j\,\alpha_{n3} - \beta_{n3} \;\therefore\; \alpha_{n3} > 0 \;\therefore\; \text{Attenuation} \\
j\,\gamma_{n4} = -\,\delta_n \sqrt{1 + w_n^2 - \sqrt{1 + 4\,w_n^2}} - \dfrac{k_{01} + k_{02}}{2} = \\
\qquad = j\,\alpha_{n4} - \beta_{n4} \;\therefore\; \alpha_{n4} < 0 \;\therefore\; \text{Amplification}
\end{array}\right. \qquad (56)$$

Neither amplification nor attenuation

The foregoing reveals that four different waves travel in the electron beam which we have described. However, amplification is present only with one of these waves, which is determined by the propagation constant $\gamma_{n4} = \alpha_{n4} + j\,\beta_{n4}$. For this wave, there is

$$\left\{\begin{array}{l}
\alpha_{n4} = -\,\delta_n \sqrt{\sqrt{1 + 4\,w_n^2} - (1 + w_n^2)} \quad ; \quad w_n^2 < 2 \\
\beta_{n4} = \quad \dfrac{k_{01} + k_{02}}{2}
\end{array}\right. \qquad (57)$$

It should be noted that the procedure employed in this consideration is fully analogous to that used in circular cross section electron wave tubes.

Beam waves obtaining when two electron beams are completely mixed

Beam No. 1;

Considering the notations employed, and formula (12), we obtain the following expression for the axial component of the electric field:

$$\left\{\begin{array}{l}
E_{1zn}^{(e,0)} = \sum\limits_{i=1}^{4} B_{1ni}^{(e,0)} \{Se_l\,(c_1, \cos v)\; Je_l\,(c_1, \cosh \zeta) + \\
\qquad p_{1ni}^{(e,0)}\; So_l\,(c_1, \cos v)\; Jo_l\,(c_1, \cosh \zeta)\}\, e^{j\omega t - \gamma_{ni}^{(e,0)} \cdot x}
\end{array}\right. \qquad (58)$$

Since the wave breaks up into odd and even waves, we shall write separate formulas for both. Thus, we obtain

$$\begin{cases} E^{(e)}_{1xn} \text{ for even waves} \\ \qquad E^{(e)}_{1xn} = Se_l\,(c_1, \cos v)\, Je_l\,(c_1, \cosh \zeta) \cdot e^{j\omega t} \sum_{i=1}^{4} B^{(e)}_{1ni} \cdot e^{-\gamma^{(e)}_{ni} \cdot x} \\ E^{(0)}_{1xn} \text{ for odd waves} \\ \qquad E^{(0)}_{1xn} = So_l\,(c_1, \cos v)\, Jo_l\,(c_1, \cosh \zeta) \cdot e^{j\omega t} \sum_{i=1}^{4} B^{(0)}_{1ni} \cdot e^{-\gamma^{(0)}_{ni} \cdot x} \end{cases} \tag{59}$$

Similarly, the expressions of v and J_x acquire the form

$$\begin{cases} v^{(e)}_{1n} = + j \dfrac{e}{\omega m} \cdot k_{01}\, Se_l\,(c_1, \cos v)\, Je_l\,(c_1, \cosh \zeta) \cdot e^{j\omega t} \sum_{i=1}^{4} \dfrac{B^{(e)}_{1ni}}{\triangle \gamma^{(e)}_{1ni}} \cdot e^{-\gamma^{(e)}_{ni} \cdot x} \\ v^{(0)}_{1n} = + j \dfrac{e}{\omega m} \cdot k_{01}\, So_l\,(c_1, \cos v)\, Jo_l\,(c_1, \cosh \zeta) \cdot e^{j\omega t} \sum_{i=1}^{4} \dfrac{B^{(0)}_{1ni}}{\triangle \gamma^{(0)}_{1ni}} \cdot e^{-\gamma^{(0)}_{ni} \cdot x} \end{cases} \tag{60}$$

$$\begin{cases} J^{(e)}_{1xn} = - j \dfrac{k_{01}}{2\, z_{01}}\, Se_l\,(c_1, \cos v)\, Je_l\,(c_1, \cosh \zeta) \cdot e^{j\omega t} \sum_{i=1}^{4} \dfrac{B^{(e)}_{1ni}}{[\triangle \gamma^{(e)}_{1ni}]^2} \cdot e^{-\gamma^{(e)}_{ni} \cdot x} \\ J^{(0)}_{1xn} = - j \dfrac{k_{01}}{2\, z_{01}}\, So_l\,(c_1, \cos v)\, Jo_l\,(c_1, \cosh \zeta) \cdot e^{j\omega t} \sum_{1=i}^{4} \dfrac{B^{(0)}_{1ni}}{[\triangle \gamma^{(0)}_{1ni}]^2} \cdot e^{-\gamma^{(0)}_{ni} \cdot x} \end{cases} \tag{61}$$

Beam No. 2:

The expressions of E_x, v and J_x are found in analogy to the foregoing. They are:

$$\begin{cases} E^{(e)}_{2xn} = Se_l\ (c_2, \cos v)\ Je_l\ (c_2, \cosh \zeta) \cdot e^{j\omega t} \sum_{i=1}^{4} B^{(e)}_{2ni} \cdot e^{-\gamma^{(e)}_{ni} \cdot x} \\ E^{(0)}_{2xn} = So_l\ (c_2, \cos v)\ Jo_l\ (c_2, \cosh \zeta) \cdot e^{j\omega t} \sum_{i=1}^{4} B^{(0)}_{2ni} \cdot e^{-\gamma^{(0)}_{ni} \cdot x} \end{cases} \tag{62}$$

$$\begin{cases} v^{(e)}_{2n} = + j \dfrac{e}{\omega m}\, k_{02} \cdot Se_l\,(c_2, \cos v)\, Je_l\,(c_2, \cosh \zeta) \cdot e^{j\omega t} \sum_{i=1}^{4} \dfrac{B^{(e)}_{2ni}}{\triangle \gamma^{(e)}_{2ni}} \cdot e^{-\gamma^{(e)}_{ni} \cdot x} \\ v^{(0)}_{2n} = + j \dfrac{e}{\omega m}\, k_{02} \cdot So_l\,(c_2, \cos v)\, Jo_l\,(c_2, \cosh \zeta) \cdot e^{j\omega t} \sum_{i=1}^{4} \dfrac{B^{(0)}_{2ni}}{\triangle \gamma^{(0)}_{2ni}} \cdot e^{-\gamma^{(0)}_{ni} \cdot x} \end{cases} \tag{63}$$

$$\begin{cases} J^{(e)}_{2xn} = -j\,\dfrac{k_{01}}{2\,z_{02}}\,Se_l\,(c_2,\cos v)\,Je_l\,(c_2,\cosh\zeta)\cdot e^{j\omega t}\displaystyle\sum_{i=1}^{4}\frac{B^{(e)}_{2ni}}{[\triangle\,\gamma^{(e)}_{2ni}]^2}\cdot e^{-\gamma^{(e)}_{ni}\cdot x} \\ J^{(0)}_{2xn} = -j\,\dfrac{k_{02}}{2\,z_{02}}\,So_l\,(c_2,\cos v)\,Jo_l\,(c_2,\cosh\zeta)\cdot e^{j\omega t}\displaystyle\sum_{i=1}^{4}\frac{B^{(0)}_{2ni}}{[\triangle\,\gamma^{(0)}_{2ni}]^2}\cdot e^{-\gamma^{(0)}_{ni}\cdot x} \end{cases} \tag{64}$$

It should be noted with regard to the indices that the first subindex (1 or 2) of E, v and B refers to the beam, while the upper index (e *or* o) indicates an even or odd wave. Moreover,

$$k_{01} = \frac{\omega}{v_{01}} \qquad \text{and} \qquad k_{02} = \frac{\omega}{v_{02}}$$

$$z_{01} = \frac{U_{01}}{J_{01}} \qquad \text{and} \qquad z_{02} = \frac{U_{02}}{J_{02}}$$

With respect to the notations one should keep in mind that

$$c_1 = \Lambda_n\,c = j\,D_n\,c \quad , \quad H_n = h_0\,D_n$$

$$c^2 = q^2\,(k_0^2 + \gamma_n^2) \quad , \quad h_0^2 = -(\gamma_n^2 + k_0^2) = \beta_n^2 - k_0^2, \text{ where } \alpha_n = 0$$

$$\Lambda_n^2 = \left[1 - \frac{a^2}{(k_1 + j\,\gamma_n)^2}\right] \quad , \quad D_n^2 = \left[\frac{a^2}{(k_1 + j\,\gamma_n)^2} - 1\right]$$

On the other hand,

$$\frac{a^2}{(k_1 + j\,\gamma_n)^2} = \left(\frac{H_n}{h_0}\right)^2 + 1 \tag{65}$$

It should be noted furthermore that γ_n occurring in the expression of c represents the propagation constant of a wave travelling in free space and of the wave travelling simultaneously in one electron beam. It is justified in this instance to assume that $\alpha_n = 0$. As to the expressions for Λ_n or D_n, we observe that the member $\dfrac{a^2}{(k_1 + j\,\gamma_n)^2}$ occurring in these expressions is obtained from equation (65). It can be seen that the expressions for c_1 and c contain the propagation factor γ_n. In the expressions of Se_l, Je_l, So_l and Jo_l we use, however, as a good approximation $\gamma_n = +\,j\,\beta_n \cong j\,k_1 = j\,\dfrac{\omega}{v_0}$ = constant, for chosen values of ω and v_0.

All that has been said above has relation to the case where only one electron beam is considered. Giving our attention to two completely mixed electron beams, we started, in order to find the value of the propagation constant, with the equation

$$D_n^2 = \left(\frac{H_n}{h_0}\right)^2 = \left[\frac{a^2/2}{(k_{01} + j\,\gamma_n)^2} + \frac{a^2/2}{(k_{02} + j\,\gamma_n)^2} - 1\right].$$

From this equation the value γ_{ni} was found for the propagation constant. This value should thus be employed whenever the combined effect of two completely mixed electron beams is studied. It should thus be noted that also c_1 and c_2 in the expressions of Se_l and Je_l or of So_l and Jo_l are functions of γ_{ni} but as a good approximation we use here $\gamma_{ni} \cong j\,\beta_{ni} \cong \frac{1}{2}\,j\,(k_{01} + k_{02}) = \frac{j\omega}{2}\left(\frac{1}{v_{01}} + \frac{1}{v_{02}}\right) =$ constant, for chosen values of ω, v_{01} and v_{02}. This is essential in simplifying calculations. Furthermore, the following has to be kept in mind with regard to the notations:

$$c_1 = A_{1n} \cdot c \quad , \quad A_{1n}^2 = \left[1 - \frac{a^2/2}{(k_{01} + j\,\gamma_{ni})^2}\right]$$

$$c_2 = A_{2n} \cdot c \quad , \quad A_{2n}^2 = \left[1 - \frac{a^2/2}{(k_{02} + j\,\gamma_{ni})^2}\right]$$

$$c^2 = q^2\,(k_0^2 + \gamma_{ni}^2), \quad k_0 = \frac{\omega}{c_0}, \quad |\alpha_{ni}| \ll |\beta_{ni}|, \quad \text{for } i = 1 - 4.$$

From (56) we see that $\alpha_{n1} = \alpha_{n2} = 0$.

Boundary conditions for two completely mixed electron beams

The boundary conditions are obtained by examining the expressions for velocity and total current in both beams. According to the foregoing, the following equation is valid at the time $t = 0$ and in the point $x = 0$ when both electron beams are modulated:

$$\begin{cases} \{v_{1n}^{(e,0)}\}_{\substack{x=0\\t=0}} = \hat{v} \quad ; \quad \{v_{2n}^{(e,0)}\}_{\substack{x=0\\t=0}} = \hat{v} \\ \{I_{1xn}^{(e,0)}\}_{\substack{x=0\\t=0}} = 0 \quad ; \quad \{I_{2xn}^{(e,0)}\}_{\substack{x=0\\t=0}} = 0 \end{cases} \tag{66}$$

It should be noted with regard to the notations that

v_1 = *A.C.* component of electron velocity in beam No. 1
v_2 = » » » » » » » No. 2

The current I_x is obtained from the current density J_x by integration over the beam cross section:

$$I_{1xn} = \int_0^{\zeta_1} \int_0^{2\pi} J_{1xn} \left\{q^2 \cosh^2 \zeta \left[1 - \left(\frac{\cos v}{\cosh \zeta}\right)^2\right] dv\right\} d\zeta$$

Study of the expression of velocity shows that it is a function of place in the electron beam. The coordinates expressing the place can be eliminated if the maximum value $\hat{v}$ of the alternating component is assumed to be constant all over the cross section. The problem will thus be to integrate the expressions over the cross section of the electron beam.

The boundary equation (66) can now be presented, for even waves, in the following form:

$$\begin{cases} + j \dfrac{e}{\omega m} k_{01} \cdot Se_l(c_1, \cos v)\, Je_l(c_1, \cosh \zeta) \displaystyle\sum_{i=1}^{4} \frac{B_{1ni}^{(e)}}{\triangle \gamma_{1ni}^{(e)}} = \hat{v} \\ \displaystyle\sum_{i=1}^{4} \frac{B_{1ni}^{(e)}}{[\triangle \gamma_{1ni}^{(e)}]^2} = 0 \\ + j \dfrac{e}{\omega m} \cdot k_{02} \cdot Se_l(c_2, \cos v)\, Je_l(c_2, \cosh \zeta) \displaystyle\sum_{i=1}^{4} \frac{B_{1ni}^{(e)}}{\triangle \gamma_{2ni}^{(e)}} = \hat{v} \\ \displaystyle\sum_{i=1}^{4} \frac{B_{2ni}^{(e)}}{[\triangle \gamma_{2ni}^{(e)}]^2} = 0 \end{cases} \tag{67}$$

It is obvious that analogous expressions are obtained for odd waves.

Hence onward we shall only consider even waves, for which reason we omit the upper index (e). The above-mentioned boundary condition equation is now written

$$\begin{cases} \displaystyle\sum_{i=1}^{4} \frac{B_{1ni}}{\triangle \gamma_{1ni}} = \frac{B_1}{k_{01}} = \\ \dfrac{\hat{v} \cdot a_{1\,\mathrm{ell}}}{j \dfrac{e}{\omega m} \cdot k_{01} \displaystyle\int_0^{\zeta_1} \int_0^{2\pi} Se_l(c_1, \cos v)\, Je_l(c_1, \cosh \zeta) \left\{ q^2 \cosh^2 \zeta \left[1 - \left(\frac{\cos v}{\cosh \zeta} \right)^2 \right] \right\} dv\, d\zeta} \\ \displaystyle\sum_{i=1}^{4} \frac{B_{1ni}}{(\triangle \gamma_{1ni})^2} = 0 \\ \displaystyle\sum_{i=1}^{4} \frac{B_{2ni}}{\triangle \gamma_{2ni}} = \frac{B_2}{k_{02}} = \\ \dfrac{\hat{v} \cdot a_{1\,\mathrm{ell}}}{j \dfrac{e}{\omega m} \cdot k_{02} \displaystyle\int_0^{\zeta_1} \int_0^{2\pi} Se_l(c_2, \cos v)\, Je_l(c_2, \cosh \zeta) \left\{ q^2 \cosh^2 \zeta \left[1 - \left(\frac{\cos v}{\cosh \zeta} \right)^2 \right] \right\} dv\, d\zeta} \\ \displaystyle\sum_{i=1}^{4} \frac{B_{2ni}}{(\triangle \gamma_{2ni})^2} = 0 \end{cases} \tag{68}$$

It should be noted that

$$\left.\begin{aligned}\Delta\gamma_{1ni} &= \Delta\gamma_{0ni} + \Delta k\\ \Delta\gamma_{2ni} &= \Delta\gamma_{0ni} - \Delta k\end{aligned}\right\} \text{ and } \Delta\gamma_{0n1} = -\Delta\gamma_{0n2}\ ,\ \Delta\gamma_{0n3} = -\Delta\gamma_{0n4}$$

$$\therefore \Delta\gamma_{1n1} = -\Delta\gamma_{2n2}\ ,\ \Delta\gamma_{1n2} = -\Delta\gamma_{2n1}\ ,\ \Delta\gamma_{1n3} = -\Delta\gamma_{2n4}\ ,$$

$$\Delta\gamma_{1n4} = -\Delta\gamma_{2n3}$$

At this stage some simplifying assumptions are indicated. Let us assume that $B_{1ni} = B_{2ni} = B_{ni}$, $k_{01} \cong k_{02}$ and

$$c_1 \cong c_2 \qquad \therefore \frac{B_1}{k_{01}} = \frac{B_2}{k_{02}} = B$$

Thus, we obtain

$$\left\{\begin{aligned}&\frac{B_{n1}}{\Delta\gamma_{1n1}} + \frac{B_{n2}}{\Delta\gamma_{1n2}} + \frac{B_{n3}}{\Delta\gamma_{1n3}} + \frac{B_{n4}}{\Delta\gamma_{1n4}} = B\\ &\frac{B_{n1}}{(\Delta\gamma_{1n1})^2} + \frac{B_{n2}}{(\Delta\gamma_{1n2})^2} + \frac{B_{n3}}{(\Delta\gamma_{1n3})^2} + \frac{B_{n4}}{(\Delta\gamma_{1n4})^2} = 0\\ &\frac{B_{n1}}{\Delta\gamma_{1n2}} + \frac{B_{n2}}{\Delta\gamma_{1n1}} + \frac{B_{n3}}{\Delta\gamma_{1n4}} + \frac{B_{n4}}{\Delta\gamma_{1n3}} = -B\\ &\frac{B_{n1}}{(\Delta\gamma_{1n2})^2} + \frac{B_{n2}}{(\Delta\gamma_{1n1})^2} + \frac{B_{n3}}{(\Delta\gamma_{1n4})^2} + \frac{B_{n4}}{(\Delta\gamma_{1n3})^2} = 0\end{aligned}\right. \tag{69}$$

Let be $B_{n1} = x$, $B_{n2} = y$, $B_{n3} = z$, $B_{n4} = u$

$\Delta\gamma_{1n1} = a$, $\Delta\gamma_{1n2} = b$, $\Delta\gamma_{1n3} = c$, $\Delta\gamma_{1n4} = d$

Then $a + b = c + d$

Consequently

$$\left\{\begin{aligned}&\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + \frac{u}{d} = +B\\ &\frac{x}{a^2} + \frac{y}{b^2} + \frac{z}{c^2} + \frac{u}{d^2} = 0\\ &\frac{x}{b} + \frac{y}{a} + \frac{z}{d} + \frac{u}{c} = -B\\ &\frac{x}{b^2} + \frac{y}{a^2} + \frac{z}{d^2} + \frac{u}{c^2} = 0\end{aligned}\right. \tag{70}$$

We find that $y = -x$ and $u = -z$

Hence

$$\left\{\begin{aligned} x &= +B\,\frac{1}{b-a}\cdot\frac{a^2b^2}{\left(ab-cd\,\dfrac{a+b}{c+d}\right)} \\ &= +B\,\frac{\delta_n\,(\sqrt{1+4\,w_n^2}+1)^2}{4\,\sqrt{(1+4\,w_n^2)\,(1+w_n^2+\sqrt{1+4\,w_n^2})}} = -y \\ z &= -B\,\frac{1}{d-c}\cdot\frac{c^2d^2}{\left(ab\,\dfrac{c+d}{a+b}-cd\right)} \\ &= -B\,\frac{\delta_n\,(\sqrt{1+4\,w_n^2}-1)^2}{4\,\sqrt{(1+4\,w_n^2)\,(1+w_n^2-\sqrt{1+4\,w_n^2})}} = -u \end{aligned}\right. \tag{71}$$

It should be noted that the result is analogous to that obtained for circular cross section electron wave tubes.

Total current density and total current

The total current density is obviously the sum of the current densities of electron beam No. 1 and of electron beam No. 2. We may thus write

$$Jx_{nT} = Jx_{1n} + Jx_{2n} \tag{72}$$

The total current is obtained by integration of the current density over the cross section:

$$Ix_{nT} = \int_0^{\zeta_1}\int_0^{2\pi} Jx_{nT}\left\{q^2\cosh^2\zeta\left[1-\left(\frac{\cos v}{\cosh\zeta}\right)^2\right]\right\}dv\,d\zeta \tag{73}$$

In the following only even waves will be considered. It is evident that completely analogous expressions are obtained for odd waves. Considering the notations employed, Jx_{nT} can be written

$$\left\{\begin{aligned} Jx_{nT} = &-j\,\frac{k_{01}}{2\,z_{01}}\,Se_l\,(c_1,\cos v)\,Je_l\,(c_1,\cosh\zeta)\cdot e^{j\omega t}\sum_{i=1}^{4}\frac{B_{1ni}}{(\triangle\gamma_{1ni})^2}\,e^{-\gamma_{ni}\cdot x} \\ &-j\,\frac{k_{02}}{2\,z_{02}}\,Se_l\,(c_2,\cos v)\,Je_l\,(c_2,\cosh\zeta)\cdot e^{j\omega t}\sum_{i=1}^{4}\frac{B_{2ni}}{(\triangle\gamma_{2ni})^2}\,e^{-\gamma_{ni}\cdot x} \end{aligned}\right. \tag{74}$$

We assume that $B_{1ni} = B_{2ni} = B_{ni}$, $a_1^2 = a_2^2 = \frac{a^2}{2}$, $c_1 \cong c_2$ and $k_{01} \cong k_{02}$

It will be shown in Appendix 3 that $\frac{k_{01}}{z_{01}} = \frac{k_{02}}{z_{02}} = \frac{k_0}{z_0}$. It should be noted with regard to the notations that

$$B_{n1} = -B_{n2}, \qquad B_{n3} = -B_{n4}$$
$$\triangle\gamma_{1n1} = -\triangle\gamma_{2n2},\ \triangle\gamma_{1n2} = -\triangle\gamma_{2n1},\ \triangle\gamma_{1n3} = -\triangle\gamma_{2n4},\ \triangle\gamma_{1n4} = -\triangle\gamma_{2n3}$$

Consequently, the expression of the total current density may be written.

$$\begin{cases} Jx_{n\,T} = -j\,\dfrac{k_0}{z_0}\,Se_l\,(c_1,\cos v)\,Je_l\,(c_1,\cosh\zeta)\cdot e^{j\omega t}\cdot \\ \quad \cdot\left\{\dfrac{B_{n1}}{(\triangle\gamma_{1n1})^2}\,\dfrac{1}{2}\,(e^{-\gamma_{n1}\cdot x} - e^{-\gamma_{n2}\cdot x}) + \dfrac{B_{n1}}{(\triangle\gamma_{1n2})^2}\cdot\dfrac{1}{2}\,(e^{-\gamma_{n1}\cdot x} - e^{-\gamma_{n2}\cdot x})\right. \\ \quad + \dfrac{B_{n3}}{(\triangle\gamma_{1n3})^2}\cdot\dfrac{1}{2}\,(e^{-\gamma_{n3}\cdot x} - e^{-\gamma_{n4}\cdot x}) \\ \quad \left. + \dfrac{B_{n3}}{(\triangle\gamma_{1n4})^2}\cdot\dfrac{1}{2}\,(e^{-\gamma_{n3}\cdot x} - e^{-\gamma_{n4}\cdot x})\right\} \end{cases} \tag{75}$$

The expression just obtained can be written in a slightly different form considering that

$$\begin{cases} -\gamma_{n1} = +j\left\{-k_0 + \delta_n\sqrt{1 + w_n^2 + \sqrt{1 + 4\,w_n^2}}\right\}, & k_0 = \dfrac{k_{01} + k_{02}}{2} \\ -\gamma_{n2} = +j\left\{-k_0 - \delta_n\sqrt{1 + w_n^2 + \sqrt{1 + 4\,w_n^2}}\right\}, & \delta_n = \dfrac{\triangle k}{w_n} \\ -\gamma_{n3} = +j\left\{-k_0 + \delta_n\sqrt{1 + w_n^2 - \sqrt{1 + 4\,w_n^2}}\right\} & \\ -\gamma_{n4} = +j\left\{-k_0 - \delta_n\sqrt{1 + w_n^2 - \sqrt{1 + 4\,w_n^2}}\right\} & \end{cases} \tag{76}$$

In the interaction region there is further

$$\begin{cases} -\gamma_{n3} = -j\,k_0 - \delta_n\sqrt{\sqrt{1 + 4\,w_n^2} - (1 + w_n^2)} \\ -\gamma_{n4} = -j\,k_0 + \delta_n\sqrt{\sqrt{1 + 4\,w_n^2} - (1 + w_n^2)} \end{cases}$$

Consequently,

$$\frac{1}{2}\,(e^{-\gamma_{n1}\cdot x} - e^{-\gamma_{n2}\cdot x}) = e^{-jk_0\cdot x}\cdot j\sin\left(\delta_n\cdot x\cdot\sqrt{\sqrt{1 + 4\,w_n^2} + (1 + w_n^2)}\right)$$

$$\frac{1}{2}\,(e^{-\gamma_{n3}\cdot x} - e^{-\gamma_{n4}\cdot x}) = -e^{-jk_0\cdot x}\cdot\sinh\left(\delta_n\cdot x\cdot\sqrt{\sqrt{1 + 4\,w_n^2} - (1 + w_n^2)}\right)$$

J_{xnT} will thus have the form

$$\left\{\begin{array}{l} J_{xnT} = e^{-jk_0 \cdot x + j\omega t}\, Se_l\,(c_1, \cos v)\, Je_l\,(c_1, \cosh \zeta)\, \dfrac{k_0}{z_0} \cdot \\ \left\{B_{n1} \sin\left(\delta_n \cdot x \cdot \sqrt{\sqrt{1 + 4\,w_n^2} + (1 + w_n^2)}\right) \left[\dfrac{1}{(\triangle\, \gamma_{1n1})^2} + \dfrac{1}{(\triangle\, \gamma_{1n2})^2}\right]\right. \\ \left. + j\, B_{n3} \sinh\left(\delta_n \cdot x \cdot \sqrt{\sqrt{1 + 4\,w_n^2} - (1 + w_n^2)}\right) \left[\dfrac{1}{(\triangle\, \gamma_{1n3})^2} + \dfrac{1}{(\triangle\, \gamma_{1n4})^2}\right]\right\} \end{array}\right. \tag{77}$$

Taking into account the employed notations, we have

$$\frac{1}{(\triangle\, \gamma_{1n1})^2} + \frac{1}{(\triangle\, \gamma_{1n2})^2} = \frac{1}{a^2} + \frac{1}{b^2} = \frac{a^2 + b^2}{a^2\, b^2}\,, \quad B_{n1} = + \frac{B}{b - a} \cdot \frac{a^2 b^2}{ab - cd}$$

$$\frac{1}{(\triangle\, \gamma_{1n3})^2} + \frac{1}{(\triangle\, \gamma_{1n4})^2} = \frac{1}{c^2} + \frac{1}{d^2} = \frac{c^2 + d^2}{c^2\, d^2}\,, \quad B_{n3} = - \frac{B}{d - c} \cdot \frac{c^2 d^2}{ab - cd}$$

Moreover, there is

$$a^2 + b^2 = 2\,\delta_n^2\,(1 + 2\,w_n^2 + \sqrt{1 + 4\,w_n^2}) = \delta_n^2\,(1 + \sqrt{1 + 4\,w_n^2})^2$$

$$c^2 + d^2 = 2\,\delta_n^2\,(1 + 2\,w_n^2 - \sqrt{1 + 4\,w_n^2}) = \delta_n^2\,(1 - \sqrt{1 + 4\,w_n^2})^2$$

The expression (77) can thus be written

$$\left\{\begin{array}{l} J_{xnT} = B \cdot e^{-jk_0 \cdot x + j\omega t}\, Se_l\,(c_1, \cos v)\, Je_l\,(c_1, \cosh \zeta) \cdot \\ \dfrac{k_0}{z_0}\left\{\dfrac{1 + 2\,w_n^2 + \sqrt{1 + 4\,w_n^2}}{2\,\delta_n \sqrt{(1 + 4\,w_n^2)\,(1 + w_n^2 + \sqrt{1 + 4\,w_n^2})}} \sin\left(x \cdot \delta_n \sqrt{\sqrt{1 + 4\,w_n^2} + (1 + w_n^2)}\right)\right. \\ \left. + \dfrac{\sqrt{1 + 4\,w_n^2} - (1 + 2\,w_n^2)}{2\,\delta_n \sqrt{[\sqrt{1 + 4\,w_n^2} - (1 + w_n^2)]\,(1 + 4\,w_n^2)}} \sinh\left(x \cdot \delta_n \sqrt{\sqrt{1 + 4\,w_n^2} - (1 + w_n^2)}\right)\right\} \\ J_{xnT} = B\, e^{-jk_0 \cdot x + j\omega t}\, Se_l\,(c_1, \cos v)\, Je_l\,(c_1, \cosh \zeta)\, \dfrac{k_0}{z_0} \cdot \\ \left\{\dfrac{(1 + \sqrt{1 + 4\,w_n^2})^2}{4\sqrt{1 + 4\,w_n^2}} \cdot \dfrac{\sin\left(x\,\delta_n \sqrt{\sqrt{1 + 4\,w_n^2} + 1 + w_n^2}\right)}{\delta_n \sqrt{\sqrt{1 + 4\,w_n^2} + 1 + w_n^2}}\right. \\ \left. - \dfrac{(\sqrt{1 + 4\,w_n^2} - 1)^2}{4\sqrt{1 + 4\,w_n^2}} \cdot \dfrac{\sinh\left((x \cdot \delta_n \sqrt{\sqrt{1 + 4\,w_n^2} - (1 + w_n^2)}\right)}{\delta_n \sqrt{\sqrt{1 + 4\,w_n^2} - (1 + w_n^2)}}\right\} \end{array}\right. \tag{78}$$

With abbreviated notations the expression of the total current density becomes

$$\left\{\begin{aligned} J_{xnT} &= B\, e^{-jk_0x+j\omega t}\, Se_l\,(c_1, \cos v)\, Je_l\,(c_1, \cosh \zeta)\, \frac{k_0}{z_0} \cdot \\ &\cdot \left\{\frac{(1+\sqrt{1+4w_n^2})^2}{4\sqrt{1+4w_n^2}} \cdot \frac{\sin\,(L\cdot x)}{L} - \frac{(\sqrt{1+4w_n^2}-1)^2}{4\sqrt{1+4w_n^2}} \cdot \frac{\sinh\,(M\cdot x)}{M}\right\} \\ &= \text{Bunch current density} + \text{Coupling current density}, \end{aligned}\right. \tag{79}$$

$$\text{where} \begin{cases} L = \delta_n \sqrt{\sqrt{1+4w_n^2}+1+w_n^2} \\ M = \delta_n \sqrt{\sqrt{1+4w_n^2}-(1+w_n^2)} \end{cases}$$

It is important to notice that the total current density is composed of two components, the bunch current density and the coupling current density.

$$\text{Bunch current density} = +B \cdot e^{-jk_0\cdot x+j\omega t}\, Se_l\,(c_1, \cos v)\, Je_l\,(c_1, \cosh \zeta) \cdot \cdot \frac{k_0}{z_0} \frac{(1+\sqrt{1+4w_n^2})^2}{4\sqrt{1+4w_n^2}} \cdot \frac{\sin\,(L\cdot x)}{L}$$

$$\text{Coupling current density} = +B\, e^{-jk_0\cdot x+j(\omega t+\pi)}\, Se_l\,(c_1, \cos v)\, Je_l\,(c_1, \cosh \zeta) \cdot \cdot \frac{k_0}{z_0} \frac{(\sqrt{1+4w_n^2}-1)^2}{4\sqrt{1+4w_n^2}} \cdot \frac{\sinh\,(M\cdot x)}{M}$$

The bunch current density varies sinusoidally with distance x and the coupling current density is dependent of x according to a hyperbolic sine function. It can thus be noted that it is the coupling current density that contains the amplification factor. It is evident that n can be eliminated from the total current density by means of summing up. Thus, we obtain for the total value of the current density

$$J_{xT} = \sum_{n=-\infty}^{+\infty} J_{xnT} = 2\sum_{n=1}^{\infty} J_{xnT} \tag{80}$$

The total current is found by integration over the cross section of the beam:

$$I_{xT} = \sum_{n=-\infty}^{+\infty} \int_0^{\zeta} \int_{n=0}^{2\pi} J_{xT} \left\{q^2 \cosh^2 \zeta \left[1 - \left(\frac{\cos v}{\cosh \zeta}\right)^2\right]\right\} dv\, d\zeta \tag{81}$$

It was found in the foregoing that amplification occurs only when the propagation constant is γ_{n4}. Its value is found to be

$$\gamma_{n4} = \alpha_{n4} + j\,\beta_{n4} = -\delta_n \sqrt{\sqrt{1 + w_n^2} - (1 + w_n^2)} + j\,\frac{k_{01} + k_{02}}{2}$$

$$\therefore \begin{cases} \alpha_{n4} = -\delta_n \sqrt{\sqrt{1 + w_n^2} - (1 + w_n^2)} \\ \beta_{n4} = \dfrac{k_{01} + k_{02}}{2} \end{cases}$$

Highest gain is obviously obtained when $|\alpha_{n4}|$ achieves its maximum value. This is

$$|\alpha_{n4}|_{\max} = |\delta_n| \cdot \frac{1}{2} \quad , \quad \text{where } w_n = \frac{\triangle k}{\delta_n} = \frac{1}{2}\sqrt{3}$$

Let us then study the effect of the different modes of propagation. We have thus to determine at which values of n (1, 2, 3, $\cdots$) the highest gain is obtained and also whether all values of n are acceptable. According to the employed notations there must be $w_n \leq \sqrt{2}$ in the interaction region:

$$w_n = \frac{\triangle k}{\delta_n} = \frac{\triangle k}{\sqrt{\dfrac{a^2}{2\left[\left(\dfrac{H_n}{h_0}\right)^2 + 1\right]}}} \leq \sqrt{2} \tag{82}$$

At the change of n, the quantities $\triangle k$, a and h_0 remain constant. It was noted before that $H_1 < H_2 < H_3 < \cdots$ Consequently, $\delta_1 > \delta_2 > \delta_3 > \cdots$ It follows immediately that $w_1 < w_2 < w_3 < \cdots$ We find that w_n increases with increasing n. Evidently, then, there is danger of exceeding the value $\sqrt{2}$ permitted in the interaction region. Let us assume that at $n = 1$ the parameters $\triangle k$, a and h_0 have been chosen in such a manner that $w_1 = \dfrac{\sqrt{3}}{2}$. According to the foregoing, $w_1 < w_2 < w_3 < \cdots$. Immediately the next value w_2 is thus in danger of exceeding the permitted value $\sqrt{2}$. This actually occurs, as has been established in the case of circular cross section tubes [4]. For this reason we restrict our study of the elliptic cross section merely to the case $n = 1$. It is true that in principle the parameters might be chosen for each value of n in such a manner that maximum gain would be obtained at this particular value. At values lower than the chosen n considerably lower gains would then obtain. For the sake of simplicity, however, we shall make here the above-mentioned restrictive assumption. We obtain thus

$$|a_{14}|_{\max} = |\delta_1| \cdot \frac{1}{2} \quad ; \quad \text{where } w_1 = \frac{1}{2}\sqrt{3}$$

$$\therefore \triangle k = \frac{1}{2}\sqrt{3} \cdot \delta_1 = \frac{1}{2}\sqrt{3}\sqrt{\frac{a^2}{2\left[\left(\frac{H_1}{h_0}\right)^2 + 1\right]}} = \frac{k_{01} - k_{02}}{2}$$

It is interesting to study the order of magnitude of $\triangle k$ at some typical values of the parameters. Let be $v_{01} = \frac{1}{10} c_0$; then $U_{01} = 2600$ volts. Let further be $\lambda = \pi$ cm and $J_{01} = J_0$ [mA/mm^2]

$$\therefore a^2 = \frac{1}{2} \cdot \frac{Z_0}{z_{01}} \cdot \frac{c_0}{v_{01}} = \frac{1}{2} \cdot \frac{120 \cdot \pi}{\frac{2600}{J_0/10}} \cdot 10 = \frac{3\pi}{130/J_0} \cong \frac{\pi}{43} \cdot J_0 \left[\frac{1}{\text{cm}^2}\right]$$

On the other hand,

$$h_0 = \sqrt{-k_0^2 - \gamma_{ni}^2} \cong \sqrt{\beta_{ni}^2 - k_0^2} \cong \sqrt{\left(\frac{\omega}{v_{01}}\right)^2 - \left(\frac{\omega}{c_0}\right)^2} \quad , \quad \begin{cases} \alpha_{ni} \cong 0 \\ \beta_{ni} \cong \dfrac{\omega}{v_{01}} \end{cases}$$

We assumed that $\frac{c_0}{v_{01}} = 10$. Now, $c_0 T = \lambda = \pi$ [cm]

$$\therefore \begin{cases} \beta_{ni} \cong \dfrac{1}{2}\left(\dfrac{\omega}{v_{01}} + \dfrac{\omega}{v_{02}}\right) \cong \dfrac{\omega}{v_{01}} = \dfrac{2\pi}{c_0 T/10} = 20 \\ \dfrac{\omega}{c_0} = k_0 = 2 \end{cases}$$

$$\therefore \quad h_0 \cong 20$$

Suppose $H_1 \cong 10$. It will be shown later that such an assumption is justified. We find thus for the value of $\triangle k$

$$\triangle k = \frac{k_{01} - k_{02}}{2} = \frac{1}{2}\sqrt{3}\sqrt{\frac{\frac{\pi}{43} \cdot J_0}{2\left[1 + \left(\frac{10}{20}\right)^2\right]}} \left[\frac{1}{\text{cm}}\right] \quad ; \quad [J_0] = \text{mA/mm}^2$$

As a typical value for J_0 we may consider 1 mA/mm^2
This yields

$$\triangle k = \frac{1}{2}\sqrt{3}\sqrt{\frac{\pi}{2 \cdot 43 \cdot 1.25}} \left[\frac{1}{\text{cm}}\right] \cong = 0{,}148 \text{ cm}^{-1}$$

which implies that

$$|v_{02} - v_{01}| \cong 0{,}0015\, c_0$$

We have thus arrived at the same result as in the case of circular cross section tubes. It is obvious that $\triangle k$ will be relatively low for typical parameter values it is required to obtain maximum gain. In practice this implies that the velocity differences $|v_{02} - v_{01}|$ between electron beams No. 1 and 2 have to be relatively small.

The case that only the first electron beam is modulated

The case presented in the foregoing can easily be extend to concern the asymmetric case, i. e., the case that only one of the beams is modulated. Assuming that beam No. 1 is modulated, we now find from the boundary conditions

$$\left\{\begin{array}{l} \{v_1\}_{\substack{x=0\\t=0}} = \hat{v} \quad ; \quad \{I_{1x}\}_{\substack{x=0\\t=0}} = 0 \\ \{v_2\}_{\substack{x=0\\t=0}} = 0 \quad ; \quad \{I_{2x}\}_{\substack{x=0\\t=0}} = 0 \end{array}\right. \tag{83}$$

Employing the previously introduced notations, we can write the expression of J_{xnT} in the following form (Appendix 4):

$$\left\{\begin{array}{l} J_{xnT} = -j\,\dfrac{k_0}{2\,z_0}\,Se_l\,(c_1,\cos v)\,Je_l\,(c_1,\cosh\zeta)\,e^{j\omega t}\cdot \\ \quad \cdot\Big\{\dfrac{(B'_{n1}+B''_{n1})\,e^{-\gamma_{n1}\cdot x}+(B'_{n1}-B''_{n1})\,e^{-\gamma_{n2}\cdot x}}{(\triangle\,\gamma_{1n1})^2} \\ \quad +\dfrac{(B'_{n1}-B''_{n1})\,e^{-\gamma_{n2}\cdot x}+(B'_{n1}+B''_{n1})\,e^{-\gamma_{n1}\cdot x}}{(\triangle\,\gamma_{1n2})^2} \\ \quad +\dfrac{(B'_{n3}+B''_{n3})\cdot e^{-\gamma_{n3}\cdot x}+(B'_{n3}-B''_{n3})\,e^{-\gamma_{n4}\cdot x}}{(\triangle\,\gamma_{1n3})^2} \\ \quad +\dfrac{(B'_{n3}-B''_{n3})\,e^{-\gamma_{n4}\cdot x}+(B'_{n3}+B''_{n3})\,e^{-\gamma_{n3}\cdot x}}{(\triangle\,\gamma_{1n4})^2}\Big\} \end{array}\right. \tag{84}$$

It should be noted that

$$\frac{1}{2}\,(e^{-\gamma_{n1}\cdot x}+e^{-\gamma_{n2}\cdot x}) = +\,e^{-j\,k_0\cdot x}\cdot\cos\left[x\cdot\delta_n\sqrt{1+w_n^2+\sqrt{1+4\,w_n^2}}\right]$$

$$\frac{1}{2}\,(e^{-\gamma_{n1}\cdot x}-e^{-\gamma_{n2}\cdot x}) = +\,e^{-j\,k_0\cdot x}\cdot j\sin\left[x\cdot\delta_n\sqrt{1+w_n^2+\sqrt{1+4\,w_n^2}}\right]$$

$$\frac{1}{2}\,(e^{-\gamma_{n3}\cdot x}+e^{-\gamma_{n4}\cdot x}) = +\,e^{-j\,k_0\cdot x}\cdot\cosh\left[x\cdot\delta_n\sqrt{\sqrt{1+4\,w_n^2}-(1+w_n^2)}\right]$$

$$\frac{1}{2}\,(e^{-\gamma_{n3}\cdot x}-e^{-\gamma_{n4}\cdot x}) = -\,e^{-j\,k_0\cdot x}\cdot\sinh\left[x\cdot\delta_n\sqrt{\sqrt{1+4\,w_n^2}-(1+w_n^2)}\right]$$

J_{xnT} can now easily be brought into the following form:

$$
\begin{aligned}
J_{xnT} = &- j \frac{k_0}{z_0} \cdot Se_l\,(c_1, \cos v)\; Je_l\,(c_1, \cosh \zeta)\; e^{j\omega t - j k_0 \cdot x} \cdot \\
&\cdot \Bigg\{ B'_{n1} \left[\frac{1}{(\triangle \gamma_{1n1})^2} + \frac{1}{(\triangle \gamma_{1n2})^2} \right] \cdot \cos \left(x \cdot \delta_n \sqrt{1 + w_n^2 + \sqrt{1 + 4 w_n^2}} \right) \\
&+ j B''_{n1} \left[\frac{1}{(\triangle \gamma_{1n1})^2} + \frac{1}{(\triangle \gamma_{1n2})^2} \right] \sin \left(x \cdot \delta_n \sqrt{1 + w_n^2 + \sqrt{1 + 4 w_n^2}} \right) \\
&+ B'_{n3} \left[\frac{1}{(\triangle \gamma_{1n3})^2} + \frac{1}{(\triangle \gamma_{1n4})^2} \right] \cosh \left(x \cdot \delta_n \sqrt{\sqrt{1 + 4 w_n^2} - (1 + w_n^2)} \right) \\
&- B''_{n3} \left[\frac{1}{(\triangle \gamma_{1n3})^2} + \frac{1}{(\triangle \gamma_{1n4})^2} \right] \sinh \left(x \cdot \delta_n \sqrt{\sqrt{1 + 4 w_n^2} - (1 + w_n^2)} \right) \Bigg\}
\end{aligned}
\tag{85}
$$

= Bunch current density + Coupling current density

$$
\begin{aligned}
\text{Bunch current density} = &- j \frac{k_0}{z_0} Se_l\,(c_1, \cos v)\; Je_l\,(c_1, \cosh \zeta)\; e^{j\omega t - j k_0 \cdot x} \cdot \\
&\cdot \frac{1}{\delta_n^2} \left[B'_{n1} \cos \left(x \cdot \delta_n \sqrt{1 + w_n^2 + \sqrt{1 + 4 w_n^2}} \right) \right. \\
&\left. + j B''_{n1} \sin \left(x \cdot \delta_n \sqrt{1 + w_n^2 + \sqrt{1 + 4 w_n^2}} \right) \right]
\end{aligned}
\tag{86}
$$

$$
\begin{aligned}
\text{Coupling current density} = &- j \frac{k_0}{z_0} Se_l\,(c_1, \cos v)\; Je_l\,(c_1, \cosh \zeta)\; e^{j\omega t - j k_0 \cdot x} \cdot \\
&\cdot \frac{1}{\delta_n^2} \left[B'_{n3} \cosh \left(x \cdot \delta_n \sqrt{\sqrt{1 + 4 w_n^2} - (1 + w_n^2)} \right) \right. \\
&\left. - B''_{n3} \sinh \left(x \cdot \delta_n \sqrt{\sqrt{1 + 4 w_n^2} - (1 + w_n^2)} \right) \right]
\end{aligned}
\tag{87}
$$

It can be concluded from the expressions just obtained that the solution is of the same form as before. Furthermore, a phase displacement is obtained in the present case. It should be noted that B'_{n1}, B''_{n1} and B'_{n3} are real but B''_{n3} is imaginary (Appendix 4).

Elliptic electron beam completely filling the elliptic tube

We have already previously, in equation (12), derived the expressions for the TM wave in the case that the electron beam completely fills the tube. We assume the conductivity σ of the tube $\zeta = \zeta_1$ to be infinite.

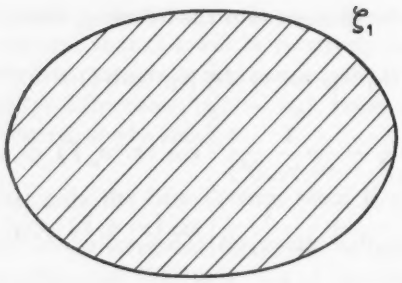


Fig. 9. Electron beam completely filling the tube.

According to the boundary conditions, the tangential component of the electric field has to vanish: $E_x = E_v = 0$ on the surface $\zeta = \zeta_1$. The expression of E_x is

$$E_x = B_1 \begin{Bmatrix} Se_l\ (c_1,\ \cos v)\ Je_l\ (c_1,\ \cosh \zeta) \\ So_l\ (c_1,\ \cos v)\ Jo_l\ (c_1,\ \cosh \zeta) \end{Bmatrix} e^{j\omega t - \gamma x} \tag{88}$$

We consider here only even waves, analogous expressions being found for odd waves. For even waves, we find

$$E_x = B_1\ Se_l\ (c_1,\ \cos v)\ Je_l\ (c_1,\ \cos \zeta)\ e^{j\omega t - \gamma x} \tag{89}$$

where

$$c_1^2 = \Lambda^2\ q^2\ (k_0^2 + \gamma^2) = \Lambda^2 \cdot c^2 \quad ; \quad \Lambda^2 = \left[1 - \frac{1}{2} \cdot \frac{Z_0}{z_0} \cdot \frac{c_0}{v_0} \cdot \frac{1}{(k_1 + j\ \gamma)^2}\right]$$

$$= \left[1 - \frac{a^2}{(k_1 + j\ \gamma)^2}\right]$$

$$k_0^2 = \left(\frac{\omega}{c_0}\right)^2 \quad ; \quad c^2 = q^2 h_0^2 = q^2\ (k_0^2 + \gamma^2) = q^2\ (k_0^2 - \beta^2) \quad ; \quad c_0^2 = \frac{1}{\mu_0\ \varepsilon_0}$$

From the boundary condition on the surface $\zeta = \zeta_1$ we obtain

$$Je_l\ (c_1,\ \cosh \zeta_1) = 0 \tag{90}$$

Since c^2 becomes negative, we introduce the notations

$$c = j\ q\ h_0 = j\ q\ (\beta^2 - k_0^2)^{1/2}$$
$$\Lambda = j\ D$$
$$c_1 = \Lambda \cdot c = -qH \quad ; \quad H = h_0\ D$$

The boundary condition equation (90) will thus have the form

$$Je_l\,(-qH,\ \cosh\zeta_1) = 0 \quad , \quad \text{or} \quad \sqrt{\frac{\pi}{2}} \sum_{n=0,1}^{\infty}{}' j^{(l-n)}\, D_n^l\, J_n\,(-qH\cdot\cosh\zeta_1) = 0$$

Considering that $J_n\,(-z) = (-1)^n\, J_n\,(z)$, we find

$$\sum_{n=0,1}^{\infty} (-1)^n \cdot j^{(l-n)}\, D_n^l\, J_n\,(qH\,\cosh\zeta_1) = 0 \tag{91}$$

It can be observed that, again, an entire series of values is obtained for H. Let us denote them with H_m and let p_m stand for the zeros of $Je_l\,(-qH,\ \cosh\zeta_1)$. Obviously

$$H_m = \frac{p_m}{q\,\cosh\zeta_1} \tag{92}$$

It should be noted that the zeros p_m are different for even and for odd values of l. Generally, we may write

$$H_m = h_0\, D_m$$

$$\therefore\ H_m = \pm\, h_0 \sqrt{\left(\frac{a}{k_1 + j\,\gamma_m}\right)^2 - 1} \tag{93}$$

Hence, we obtain

$$k_1 + j\,\gamma_m = \frac{\pm\, a}{\sqrt{\left(\frac{H_m}{h_0}\right)^2 + 1}} \tag{94}$$

In this manner we have determined the propagation constant γ_m. In principle it has the same form as before. Obviously, thus, an amplification effect would be obtained in this case, too, if we should study two simultaneous electron beams. The procedure is analogous to that presented in the foregoing.

Practical calculations ($l = 0$)

Our purpose is now to study the case where $l = 0$. We have already found before that the lowest modes for even waves are $l = 0,\ 1, \cdots$ and for odd waves $l = 1,\ 2, \cdots$

The boundary equation (28) now takes the form

$$\left\{\begin{array}{l} \dfrac{\left[\dfrac{\partial}{\partial\zeta}\, Je_0\,(-qH,\,\cosh\zeta)\right]_{\zeta=\zeta_1}}{Je_0\,(-qH,\,\cosh\zeta_1)} \\ = \dfrac{\left[\dfrac{\partial}{\partial\zeta}\, Ie_0\,(q\hbar_0,\,\cosh\zeta) - \dfrac{Ie_0\,(q\hbar_0,\,\cosh\zeta_2)}{Ke_0\,(q\hbar_0,\,\cosh\zeta_2)}\,\dfrac{\partial}{\partial\zeta}\, Ke_0\,(q\hbar_0,\,\cosh\zeta)\right]_{\zeta=\zeta_1}}{Ie_0\,(q\hbar_0,\,\cosh\zeta_1) - \dfrac{Ie_0\,(q\hbar_0,\,\cosh\zeta_2)}{Ke_0\,(q\hbar_0,\,\cosh\zeta_2)}\, Ke_0\,(q\hbar_0,\,\cosh\zeta_1)} \end{array}\right. \tag{95}$$

According to the employed notations,

$$Je_0\,(-qH,\,\cosh\zeta) = \sqrt{\frac{\pi}{2}}\cdot{\sum_{n=0}^{\infty}}'\, j^{-n}\, D_n^0\, J_n\,(-qH\,\cosh\zeta)$$

$$= \sqrt{\frac{\pi}{2}}\sum_{n=0}^{\infty}\,(+j)^n\, D_n^0\, J_n\,(qH\,\cosh\zeta)$$

It should be noted that $J_n\,(-z) = (-1)^n\, J_n\,(z)$ and $n = 0,\, 2,\, 4,\, \cdots$ Furthermore,

$$Ie_0\,(q\hbar_0,\,\cosh\zeta) = \sqrt{\frac{\pi}{2}}\cdot{\sum_{n=0}^{\infty}}'\, D_n^0\, I_n\,(q\hbar_0\,\cosh\zeta)\quad ;\quad n = 0,\, 2,\, 4,\, \cdots$$

$$Ke_0\,(q\hbar_0,\,\cosh\zeta) = -\sqrt{\frac{2}{\pi}}{\sum_{n=0}^{\infty}}'\, D_n^0\, K_n\,(q\hbar_0\,\cosh\zeta)\quad ;\quad n = 0,\, 2,\, 4,\, \cdots$$

The boundary condition equation (95) thus assumes the form

$$\left\{\begin{array}{l} qH\,\sin\zeta_1\,\dfrac{{\displaystyle\sum_{n=0}^{\infty}}'\,(+j)^n\, D_n^0\cdot J_n'\,(qH\,\cosh\zeta_1)}{{\displaystyle\sum_{n=0}^{\infty}}\,(+j)^n\, D_n^0\, J_n\,(qH\,\cosh\zeta_1)} = q\hbar_0\,\sinh\zeta_1\cdot \\ \dfrac{{\displaystyle\sum_{n=0}^{\infty}}'\, D_n^0\, K_n'\,(q\hbar_0\,\cosh\zeta_1) - \dfrac{{\displaystyle\sum_{n=0}^{\infty}}'\, D_n^0\, K_n\,(q\hbar_0\,\cosh\zeta_2)}{{\displaystyle\sum_{n=0}^{\infty}}'\, D_n^0\, I_n\,(q\hbar_0\,\cosh\zeta_2)}\,{\displaystyle\sum_{n=0}^{\infty}}'\, D_n^0\, I_n'\,(q\hbar_0\,\cosh\zeta_1)}{{\displaystyle\sum_{n=0}^{\infty}}'\, D_n^0\, K_n\,(q\hbar_0\,\cosh\zeta_1) - \dfrac{{\displaystyle\sum_{n=0}^{\infty}}'\, D_n^0\, K_n\,(q\hbar_0\,\cosh\zeta_2)}{{\displaystyle\sum_{n=0}^{\infty}}'\, D_n^0\, I_n\,(q\hbar_0\,\cosh\zeta_2)}\cdot{\displaystyle\sum_{n=0}^{\infty}}'\, D_n^0\, I_n\,(q\hbar_0\,\cosh\zeta_1)} \end{array}\right. \tag{96}$$

With abbreviated notations, it can obviously be written

$$\left\{ z \frac{\sum_{n=0}^{\infty}{}' (+j)^n D_n^0 J_n'(z)}{\sum_{n=0}^{\infty}{}' (+j)^n D_n^0 J_n(z)} \right.$$

$$\left. = u \frac{\sum_{n=0}^{\infty}{}' D_n^0 K_n'(u) - \frac{\sum_{n=0}^{\infty}{}' D_n^0 K_n(su)}{\sum_{n=0}^{\infty}{}' D_n^0 I_n(su)} \sum_{n=0}^{\infty}{}' D_n^0 I_n'(u)}{\sum_{n=0}^{\infty}{}' D_n^0 K_n(u) - \frac{\sum_{n=0}^{\infty}{}' D_n^0 K_n(su)}{\sum_{n=0}^{\infty}{}' D_n^0 I_n(su)} \sum_{n=0}^{\infty}{}' D_n^0 I_n(u)} \right. \tag{97}$$

where

$$z = qH \cosh \zeta_1 = qh_0 D \cosh \zeta_1 \quad ; \quad u = qh_0 \cosh \zeta_1$$

$$su = qh_0 \cosh \zeta_2 \quad ; \quad s = \cosh \zeta_2 / \cosh \zeta_1$$

Comparison of the elliptic and circular cross section electron wave tubes

With decreasing eccentricity the ellipse obviously approaches a circle. Thus also the boundary condition equation for the elliptic cross section has to change into that pertaining to the circular cross section electron wave tube when the eccentricity goes toward zero. We shall carry through this limit process and consider its results.

Let the eccentricity e approach zero. Now,

$$e = \frac{q}{a} = \frac{1}{\xi} = \frac{1}{\cosh \zeta} \quad \therefore \quad q \cosh \zeta = a$$

When $e \to 0$, $\therefore$ $q \to 0$ and $\cosh \zeta \to \infty$. It should be noted that $q \cosh \zeta$ remains finite.

We write $\lim_{e \to 0} z = x$ and introduce this as a new variable. Furthermore, $D_n^0 = 1$ when $c = 0$. The other D_n^0 are all zero, i. e., $D_2^0 = D_4^0 = D_6^0 = \cdots = 0$.

The left member of equation (97) thus assumes the form

$$\lim_{e\to 0} z \frac{\sum_{n=0}^{\infty}{}' (+j)^n D_n^0 J_n'(z)}{\sum_{n=0}^{\infty}{}' (+j)^n D_n^0 J_n(z)} = x \frac{J_0'(x)}{J_0(x)} = -x \frac{J_1(x)}{J_0(x)} \tag{98}$$

The right member can correspondingly be brought in to the following form, introducing $\lim_{e\to 0} u = y$ as a new variable:

$$\left\{ \begin{aligned} &\lim_{e\to 0} u \frac{\sum_{n=0}^{\infty}{}' D_n^0 K_n'(u) - \frac{\sum_{n=0}^{\infty}{}' D_n^0 K_n(su)}{\sum_{n=0}^{\infty}{}' D_n^0 I_n(su)} \sum_{n=0}^{\infty}{}' D_n^0 I_n'(u)}{\sum_{n=0}^{\infty}{}' D_n^0 K_n(u) - \frac{\sum_{n=0}^{\infty}{}' D_n^0 K_n(su)}{\sum_{n=0}^{\infty}{}' D_n^0 I_n(su)} \sum_{n=0}^{\infty}{}' D_n^0 I_n(u)} \\ &= -y \frac{K_1(y) + \frac{K_0(sy)}{I_0(sy)} I_1(y)}{K_0(y) - \frac{K_0(sy)}{I_0(sy)} I_0(y)} \quad , \text{ where } \begin{cases} I_0'(y) = +I_1(y) \\ K_0'(y) = -K_1(y) \end{cases} \end{aligned} \right. \tag{99}$$

We have thus arrived at the equation

$$x \frac{J_1(x)}{J_0(x)} = y \frac{K_1(y) + \frac{K_0(sy)}{I_0(sy)} I_1(y)}{K_0(y) - \frac{K_0(sy)}{I_0(sy)} I_1(y)} \tag{100}$$

But this is exactly the result known from the theory of the circular cross section electron wave tube. We have thus established that the boundary condition equation of the elliptic electron wave tube is transformed into the corresponding equation pertaining to the circular cross section tube when the ellipse degenerates into a circle.

Numerical computations

Our problem is to solve the equation

$$\left\{ z \frac{\sum_{n=0}^{\infty}{}' (j)^n D_n^0 J_n'(z)}{\sum_{n=0}^{\infty}{}' (j)^n D_n^0 J_n(z)} \sum_{n=0}^{\infty}{}' D_n^0 K_n'(u) - \frac{\sum_{n=0}^{\infty}{}' D_n^0 K_n(su)}{\sum_{n=0}^{\infty}{}' D_n^0 I_n(su)} \sum_{n=0}^{\infty}{}' D_n^0 I_n'(u) \right\} = u \frac{\sum_{n=0}^{\infty}{}' D_n^0 K_n(u) - \frac{\sum_{n=0}^{\infty}{}' D_n^0 K_n(su)}{\sum_{n=0}^{\infty}{}' D_n^0 I_n(su)} \sum_{n=0}^{\infty}{}' D_n^0 I_n(u)}{\sum_{n=0}^{\infty}{}' D_n^0 K_n(u) - \frac{\sum_{n=0}^{\infty}{}' D_n^0 K_n(su)}{\sum_{n=0}^{\infty}{}' D_n^0 I_n(su)} \sum_{n=0}^{\infty}{}' D_n^0 I_n(u)} \tag{101}$$

where

$$z = qh_0 \cosh \zeta_1 \; D = qH \cosh \zeta_1$$
$$u = qh_0 \cosh \zeta_1$$

Let us choose

$$\begin{aligned} \cosh \zeta_1 &= 1{,}25 \\ qh_0 &= c = 3 \text{ to } 5 \\ \cosh \zeta_2 / \cosh \zeta_1 &= s = 5 \end{aligned}$$

Furthermore,

$$h_0 = \sqrt{-k_0^2 - \gamma_{ni}^2} = \sqrt{\beta_{ni}^2 - k_0^2} \simeq \sqrt{\left(\frac{\omega}{v_{01}}\right)^2 - \left(\frac{\omega}{c_0}\right)^2} \quad ; \quad \begin{cases} \gamma_{ni} = \alpha_{ni} + j\,\beta_{ni} \\ |\alpha_{ni}| \ll |\beta_{ni}| \end{cases}$$

Let $\frac{c_0}{v_{01}} = 10$ and $c_0 T = \lambda = \pi$ cm.

We find then

$$\beta_{ni} \simeq \frac{1}{2}\left(\frac{\omega}{v_{01}} + \frac{\omega}{v_{02}}\right) \simeq \frac{\omega}{v_{01}} = \frac{2\pi}{c_0 T/10} = 20$$

$$\frac{\omega}{c_0} = k_0 = 2$$

Let $q = 0.15—0.25$ [cm] and $s = \dfrac{\cosh \zeta_2}{\cosh \zeta_1} = 5$

$\therefore\ qh_0 = c = 3.0—5.0$ [cm]

Equation (101) thus assumes the form

$$\left\{ z \frac{\sum_{n=0}^{\infty}{}' (j)^n D_n^0 J_n'(z)}{\sum_{n=0}^{\infty}{}' (j)^n D_n^0 J_n(z)} \sum_{n=0}^{\infty}{}' D_n^0 K_n'(u) - \frac{\sum_{n=0}^{\infty}{}' D_n^0 K_n(5u)}{\sum_{n=0}^{\infty}{}' D_n^0 I_n(5u)} \sum_{n=0}^{\infty}{}' D_n^0 I_n'(u) \right\} = u \frac{\sum_{n=0}^{\infty}{}' D_n^0 K_n'(u) - \frac{\sum_{n=0}^{\infty}{}' D_n^0 K_n(5u)}{\sum_{n=0}^{\infty}{}' D_n^0 I_n(5u)} \sum_{n=0}^{\infty}{}' D_n^0 I_n'(u)}{\sum_{n=0}^{\infty}{}' D_n^0 K_n(u) - \frac{\sum_{n=0}^{\infty}{}' D_n^0 K_n(5u)}{\sum_{n=0}^{\infty}{}' D_n^0 I_n(5u)} \sum_{n=0}^{\infty}{}' D_n^0 I_n(u)} \tag{102}$$

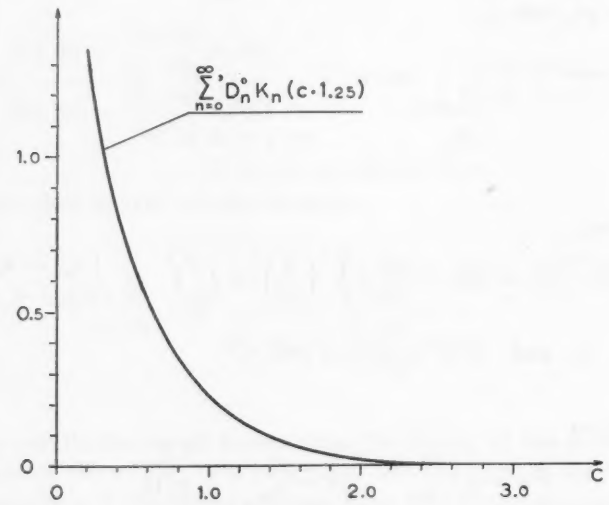


Figure 10.

In the figures 10—13, $u = c \cdot 1.25$.

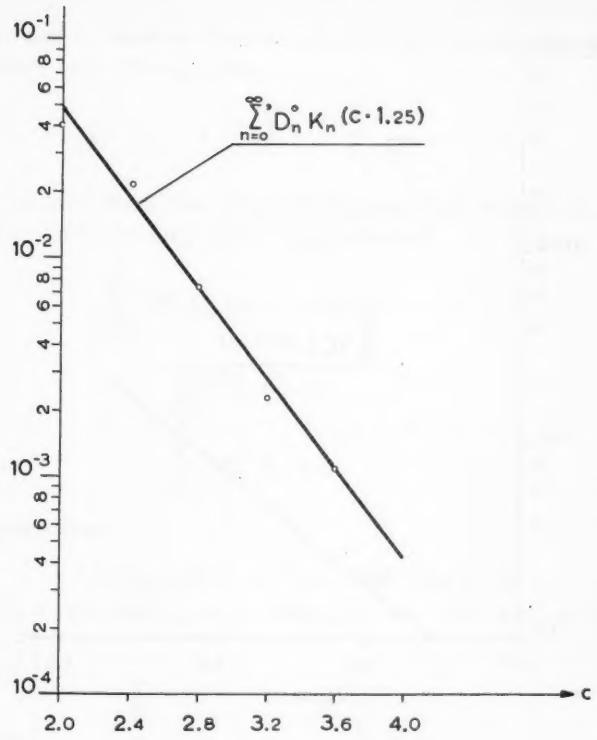


Figure 11.

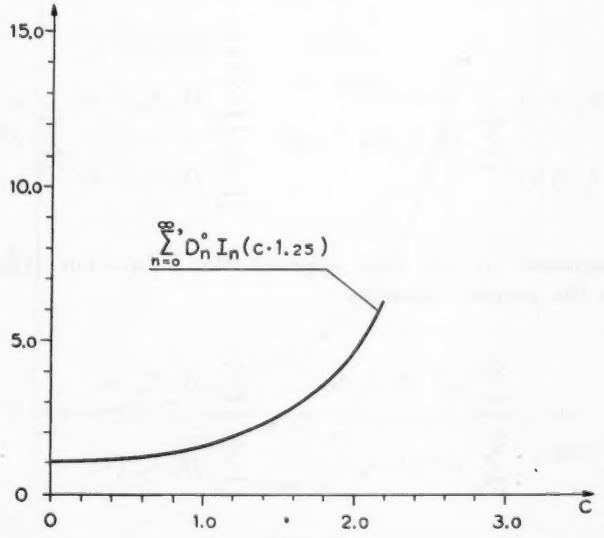


Figure 12.

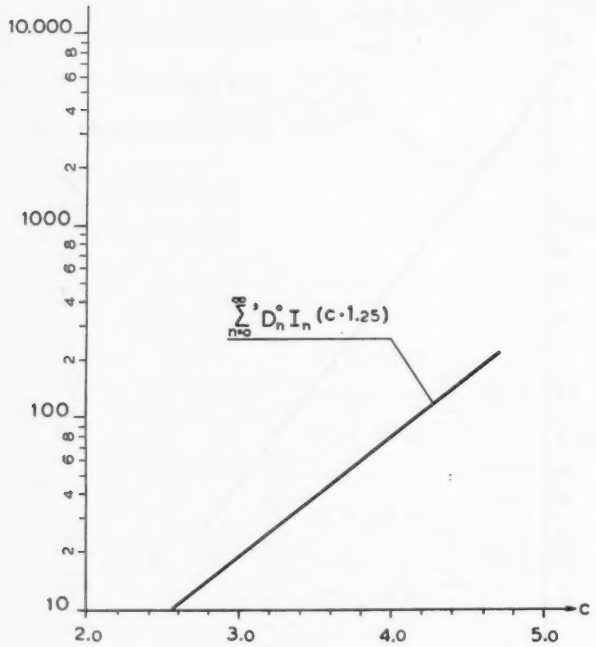


Figure 13.

It can be seen from the accompanying figures 10, 11, 12 and 13 that the terms

$$\frac{\sum_{n=0}^{\infty}{}' D_n^0 K_n (5u)}{\sum_{n=0}^{\infty}{}' D_n^0 I_n (5u)} \sum_{n=0}^{\infty}{}' D_n^0 I_n' (u) \quad \text{and} \quad \frac{\sum_{n=0}^{\infty}{}' D_n^0 K_n (5u)}{\sum_{n=0}^{\infty}{}' D_n^0 I_n (5u)} \sum_{n=0}^{\infty}{}' D_n^0 I_n (u)$$

can be neglected in the first approxination. Equation (102) is thus reduced to the simplier equation

$$z \frac{\sum_{n=0}^{\infty}{}' (j)^n D_n^0 J_n' (z)}{\sum_{n=0}^{\infty}{}' (j)^n D_n^0 J_n (z)} = u \frac{\sum_{n=0}^{\infty}{}' D_n^0 K_n' (u)}{\sum_{n=0}^{\infty}{}' D_n^0 K_n (u)} \tag{103}$$

It may be well to observe that we obtained for the circular cross section electron wave tube the equation

$$z \frac{J_0' (z)}{J_0 (z)} = y \frac{K_0' (y)}{K_0 (y)}$$

It can be seen from the obtained diagram that, when c is within the interval $2.4 \cdots 5.0$, we may write approximately

$$\sum_{n=0}^{\infty}{}' D_n^0 K_n (u) \cong \text{constant} \cdot e^{-Lu}$$

$$\therefore \quad u \frac{\sum_{n=0}^{\infty}{}' D_n^0 K_n' (u)}{\sum_{n=0}^{\infty}{}' D_n^0 K_n (u)} \cong - L \cdot u$$

Let us assume that

$$q = 0.2 \text{ cm}, \cosh \zeta_1 = 1.25 \quad \text{and} \quad q h_0 = 20$$

$$\therefore \bar{u} = q h_0 \cosh \zeta_1 = c \cdot \cosh \zeta_1 = 0.2 \cdot 20 \cdot 1.25 = 5$$

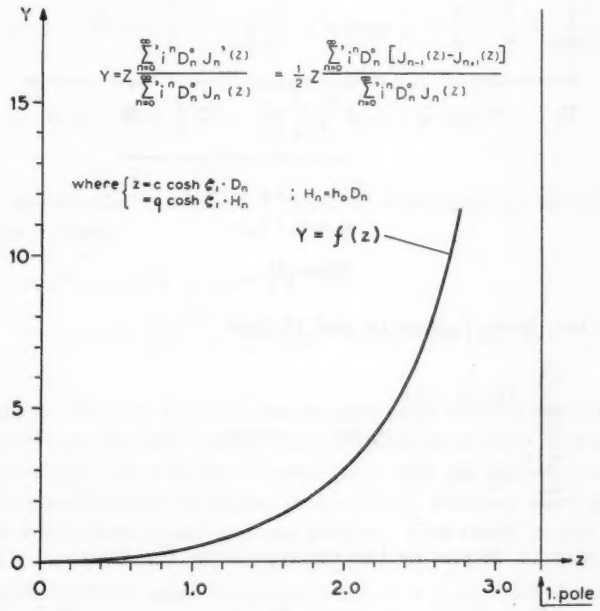
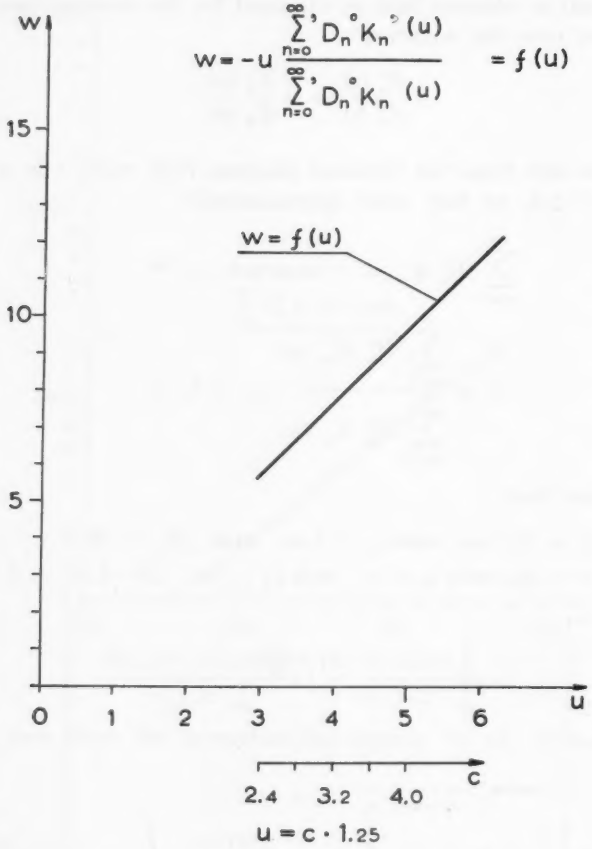


Figure 14.



$$w = -u \frac{\sum_{n=0}^{\infty}{}' D_n^0 K_n^{\circ} (u)}{\sum_{n=0}^{\infty}{}' D_n^0 K_n (u)} = f(u)$$

Figure 15.

It follows then from figures 14 and 15 that

$$-u \frac{\sum_{n=0}^{\infty}{}' D_n^0 K_n' (u)}{\sum_{n=0}^{\infty}{}' D_n^0 K_n (u)} \cong 9.7 \qquad \text{when} \qquad u = 5 \quad \therefore \quad z = 2.68$$

It is now easy to determine the corresponding H_n. It is

$$qH_n \cosh \zeta_1 = z \qquad \therefore \quad H_1 = \frac{2.68}{0.2 \cdot 1.25} \cong 10.7$$

From this, we may calculate δ_1:

$$\delta_1 = \frac{\sqrt{a^2/2}}{\sqrt{1 + \left(\frac{H_1}{h_0}\right)^2}}$$

$$a^2 = \frac{1}{2}\frac{Z_0}{z_{01}} \cdot \frac{c_0}{v_{01}} = \frac{1}{2} \cdot \frac{120 \cdot \pi}{2600\sqrt{\frac{J_0}{10}}} \cdot 10 \cong \frac{\pi}{43} J_0 \,[\mathrm{cm}^{-2}] \quad ; \quad \text{where } [J_0] = \mathrm{mA/mm^2}$$

Let us assume that $J_0 = 1$ mA/mm²

$$\therefore \sqrt{\frac{a^2}{2}} \cong 0.191 \quad \text{and} \quad \delta_1 = \frac{0.191}{\sqrt{1.286}} \cong 0.168 \,[\mathrm{cm}^{-1}]$$

It is intersting to compare this result to that pertaining to circular cross section, maintaining the cross section of the electron beam constant.

In the elliptic case, the cross section of the electron beam equals $\pi\, a\, b$. In this, $a = 0.2 \cdot 1.25 = 0.25$ cm. The value of b is obtained from the formula

$$b = a\sqrt{1 - e^2} = a\sqrt{1 - \left(\frac{q}{a}\right)^2} \quad ; \quad \text{where } e = \frac{q}{a} = \frac{1}{\xi} = \frac{1}{\cosh \zeta_1} = \frac{1}{1.25}$$

$$\therefore \pi\, a\, b = \pi \cdot 0.25 \cdot 0.25 \sqrt{1 - \left(\frac{1}{1.25}\right)^2} \cong 0.118 \text{ cm}^2$$

Let now r denote the radius of a circle of equal area as the ellipse considered above. Then,

$$\pi\, r^2 = \pi\, a\, b = \pi\, a^2 \sqrt{1 - e^2}$$

$$\therefore r = a \cdot \sqrt[4]{1 - e^2} \cong 0.25 \cdot 0.775 \cong 0.193 \text{ cm}$$

Corresponding to this, we have in the circular cross section case $H_1 \cong 10{,}0$. In the elliptic case we had $H_1 \cong 10{,}7$. This implies that δ_1, is slightly smaller in the elliptic than in the circular case, with the physical interpretation that the amplification is higher in a circular electron wave tube than in an elliptic wave tube equal in cross section. This result is just as could be expected.

Let us then consider what happens when $e \to 1$, or $\cosh \zeta_1 \to 1$. The ellipse will then change its shape towards that of a straight line. In this

transformation, let us maintain the cross section of the ellipse constant. This implies that $q \to \infty$, $a \to \infty$ and $b \to 0$. We have now to study the equation

$$z \frac{\sum_{n=0}^{\infty}{}' (j)^n D_n^0 J_n'(z)}{\sum_{n=0}^{\infty}{}' (j)^n D_n^0 J_n(z)} = u \frac{\sum_{n=0}^{\infty}{}' D_n^0 K_n'(u)}{\sum_{n=0}^{\infty}{}' D_n^0 K_n(u)} \tag{104}$$

where

$$u = qh_0 \cosh \zeta_1 = c \cdot \cosh \zeta_1$$

$$z = qH_n \cosh \zeta_1 = qh_0 D_n \cosh \zeta_1 = -c_1 \cosh \zeta_1$$

We know from the theory of Mathieu's functions that

$$Je_l(c, \cosh \psi) \to \frac{1}{\sqrt{2\pi \cdot \lambda_l}} \quad \text{when} \quad \cosh \psi \to 1 \tag{105}$$

where

$$\lambda_l = \frac{1}{\pi} \sum_{n=0}^{\infty}{}' j^{n-l} \frac{D_n^l}{D_0^l} \qquad \text{when} \quad l = \text{even}$$

Obviously thus because $J_n(-z) = J_n(z)$ for even valves of n.

$$\sum_{n=0}^{\infty}{}' (j)^n D_n^0 J_n(c_1 \cosh \zeta_1) \to \frac{1}{\sqrt{2\pi \cdot \lambda}} \quad \text{when} \quad \cosh \zeta_1 \to 1 \tag{106}$$

where

$$\lambda_0 = \frac{1}{\pi} \sum_{n=0}^{\infty}{}' (j)^n \frac{D_n^0}{D_0^0}$$

It follows immediately that

$$\sum_{n=0}^{\infty} (j)^n D_n^0 J_n'(c_1 \cdot \cosh \zeta_1) \to 0, \quad \text{when} \quad \cosh \zeta_1 \to 1 \tag{107}$$

The left member of equation (104) thus has the form

$$\text{constant} \cdot qH_n \cdot 0$$

On the other hand, the right member of the equation is found to have the form

$$\text{constant} \cdot q$$

Consequently, the product $H_n \cdot 0$ has to be finite, i. e.,

$$H_n = \infty \quad , \quad \text{where} \quad n = 1. \tag{108}$$

The result just obtained can be illustrated by means of the graphical representation of the function $H_1 = f(e)$, which has the shape shown in figure 16.

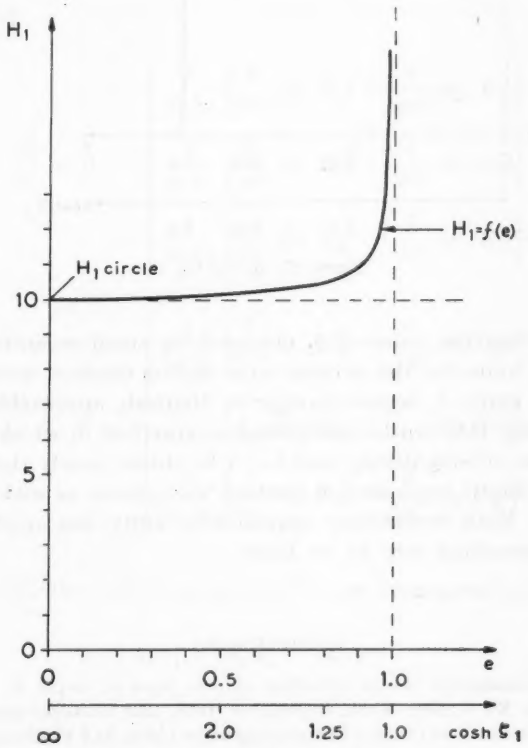


Figure 16. $H_1 = f(e)$.

On the other hand, $\delta_1 = \dfrac{\sqrt{a^2/2}}{\sqrt{1 + (H_1/h_0)^2}}$. Consequently, δ_1 can be presented as a function of the eccentricity as shown in figure 17.

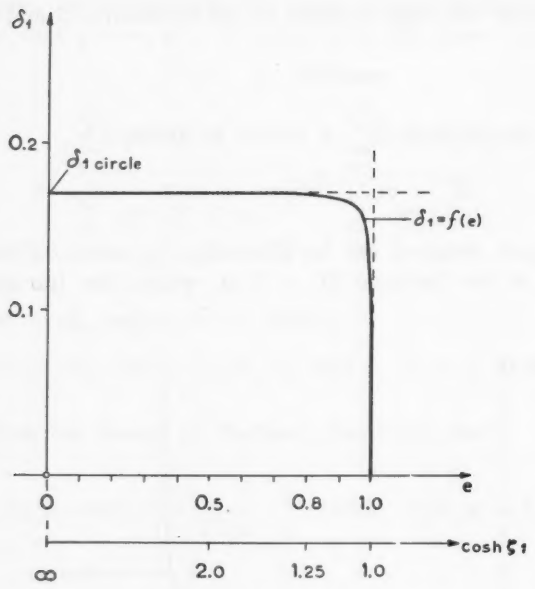


Figure 17. $\delta_1 = f(e)$.

It is seen that the value of δ_1 obtained for small eccentricities is very close to the δ_1 value for the circular cross section electron wave tube. With e approaching unity, δ_1 begins strongly to diminish, approaching zero as its limit. Physically, this can be interpreted as an effect in which it is possible at small values of eccentricity and $l = 0$ to obtain nearly the same amplification with elliptic cross section electron wave tubes as with circular cross section tubes. With eccentricity approaching unity the amplification falls abruptly, approaching zero as its limit.

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Appendix 1

In general orthogonal coordinates, the field equations can be expressed in the scalar form shown below [Stratton: Electromagnetic Theory, p. 50, First edition]:

$$\nabla \times \bar{E} + \frac{\partial \bar{B}}{\partial t} = 0 \therefore \begin{cases} \dfrac{1}{h_2 h_3}\left[\dfrac{\partial}{\partial u^2}(h_3 E_3) - \dfrac{\partial}{\partial u^3}(h_2 E_2)\right] + \dfrac{\partial B_1}{\partial t} = 0 \\ \dfrac{1}{h_3 h_1}\left[\dfrac{\partial}{\partial u^3}(h_1 E_1) - \dfrac{\partial}{\partial u^1}(h_3 E_3)\right] + \dfrac{\partial B_2}{\partial t} = 0 \\ \dfrac{1}{h_1 h_2}\left[\dfrac{\partial}{\partial u^1}(h_2 E_2) - \dfrac{\partial}{\partial u^2}(h_1 E_1)\right] + \dfrac{\partial B_3}{\partial t} = 0 \end{cases}$$

$$\nabla \times \bar{H} - \frac{\partial \bar{D}}{\partial t} = \bar{J} \therefore \begin{cases} \dfrac{1}{h_2 h_3}\left[\dfrac{\partial}{\partial u^2}(h_3 H_3) - \dfrac{\partial}{\partial u^3}(h_2 H_2)\right] - \dfrac{\partial D_1}{\partial t} = J_1 \\ \dfrac{1}{h_3 h_1}\left[\dfrac{\partial}{\partial u^3}(h_1 H_1) - \dfrac{\partial}{\partial u^1}(h_3 H_3)\right] - \dfrac{\partial D_2}{\partial t} = J_2 \\ \dfrac{1}{h_1 h_2}\left[\dfrac{\partial}{\partial u^1}(h_2 H_2) - \dfrac{\partial}{\partial u^2}(h_1 H_1)\right] - \dfrac{\partial D_3}{\partial t} = J_3 \end{cases}$$

since, generally, the curl of a vector $\bar{F}$ in an orthogonal system is

$$\nabla \times \bar{F} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \bar{i}_1 & h_2 \bar{i}_2 & h_3 \bar{i}_3 \\ \dfrac{\partial}{\partial u^1} & \dfrac{\partial}{\partial u^2} & \dfrac{\partial}{\partial u^3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}$$

In the chosen elliptic coordinates in particular, $u^1 = x$, $u^2 = \zeta$ and $u^3 = v$. Consequently, $h_i^2 = \left(\dfrac{\partial x^1}{\partial u^i}\right)^2 + \left(\dfrac{\partial x^2}{\partial u^i}\right)^2 + \left(\dfrac{\partial x^3}{\partial u^i}\right)^2$ will have the form

$$h_1 = 1$$

$$h_2 = q\sqrt{\sinh^2\zeta + \sin^2 v} \qquad \text{where } 2q = \overline{P_1 P_2}$$

$$h_3 = q\sqrt{\sinh^2\zeta + \sin^2 v} \qquad \text{because } x^1 = x = x$$

$$x^2 = y = q\cosh\zeta \cdot \cos v$$

$$x^3 = z = q\sinh\zeta \cdot \sin v$$

Let us write $q_1 = q\,(\sinh^2\zeta + \sin^2 v)^{1/2} = q\,(\cosh^2\zeta - \cos^2 v)^{1/2}$

Thus, we obtain $h_1 = 1$ and $h_2 = h_3 = q_1$.

On the basis of the foregoing, Maxwell's equations can be written in elliptic coordinates in the form shown below, assuming that the medium is characterized by the constants ε, σ and μ. We write

$$\bar{E} = Re\,(\bar{E}'\ e^{j\omega t-\gamma x}) \qquad \text{and} \qquad \bar{J} = \sigma\,\bar{E}$$

$$\bar{H} = Re\,(\bar{H}'\ e^{j\omega t-\gamma x})$$

Thus

$$\begin{cases} (\sigma + j\,\omega\,\varepsilon)\,q_1^2\,E_x = +\dfrac{\partial\,(q_1\,H_v)}{\partial\zeta} - \dfrac{\partial\,(q_1\,H_\zeta)}{\partial v} \\ (k^2 + \gamma^2)\,q_1\,E_\zeta = -\gamma\,\dfrac{\partial E_x}{\partial\zeta} - j\,\omega\,\mu\,\dfrac{\partial H_x}{\partial v} \\ (k^2 + \gamma^2)\,q_1\,E_v = -\gamma\,\dfrac{\partial E_x}{\partial v} + j\,\omega\,\mu\,\dfrac{\partial H_x}{\partial\zeta} \end{cases}$$

$$\begin{cases} -j\,\omega\,\mu\,q_1^2\,H_x = +\dfrac{\partial}{\partial\zeta}\,(q_1\,E_v) - \dfrac{\partial}{\partial v}\,(q_1\,E_\zeta) \\ (k^2 + \gamma^2)\,q_1\,H_\zeta = +(\sigma + j\,\omega\,\varepsilon)\,\dfrac{\partial E_x}{\partial v} - \gamma\,\dfrac{\partial H_x}{\partial\zeta} \\ -(k^2 + \gamma^2)\,q_1\,H_v = +(\sigma + j\,\omega\,\varepsilon)\,\dfrac{\partial E_x}{\partial\zeta} + \gamma\,\dfrac{\partial H_x}{\partial v} \end{cases}$$

where

$$k^2 = \omega^2\,\mu\,\varepsilon - j\,\omega\,\mu\,\sigma$$

$$q_1^2 = q^2\,(\sinh^2\zeta + \sin^2 v) = q^2\,(\cosh^2\zeta - \cos^2 v)$$

$$\gamma = \alpha + j\,\beta$$

We have thus obtained a system of equations, with six equations and six unknown quantities. We solve from those E_x and H_x. It is found that the obtained differential equations are similar as to their form. They have the form

$$\frac{\partial^2 F}{\partial \zeta^2} + \frac{\partial^2 F}{\partial v^2} + (k^2 + \gamma^2)\, q^2\, (\cosh^2 \zeta - \cos^2 v)\, F = 0$$

where

$$F = E_x \text{ or } H_x$$

Let us write $$F = f_1(\zeta) \cdot f_2(v)$$

Thus, we obtain

$$\frac{\partial^2 f_1}{\partial \zeta^2} = (c^2 \cosh^2 \zeta - b_l)\, f_1 = 0$$

$$\frac{\partial^2 f_2}{\partial v^2} = (b_l - c^2 \cos^2 v)\, f_2 = 0$$

where b_l = constant of separation

$$c^2 = (k^2 + \gamma^2)\, q^2 = \text{parameter}$$

In consequence of its physical character, the electromagnetic field is a univalent function of place. It follows that f_2 must be periodic with regard to the angle v, with the period 2π. It is found that periodic solutions exist only at certain characteristic values of b. Let these be denoted with b_l, where $l = 0, 1, 2, 3, \cdots$ The solutions can furthermore be divided into even and odd functions. The even functions have $l = 0, 1, 2, 3, \cdots$ and the odd functions $l = 1, 2, 3, \cdots$ The notation employed is as follows:

$f_1(\zeta)$ = radial function $= Je_l(c, \cosh \zeta)$ and $Jo_l(c, \cosh \zeta)$

$f_2(v)$ = angular function $= Se_l(c, \cos v)$ and $So_l(c, \cos v)$

where

Je ( ) and *Se* ( ) are even functions

Jo ( ) and *So* ( ) are odd functions

For the final solution we find thus

$$\left.\begin{matrix} E_x \\ H_x \end{matrix}\right\} = [B_1\, Se_l(c, \cos v)\, Je_l(c, \cosh \zeta) + B_2\, So_l(c, \cos v)\, Jo_l(c, \cosh \zeta)]\, e^{j\omega t - \gamma x}$$

or, with abbreviated notations,

$$\left.\begin{matrix} E_x \\ H_x \end{matrix}\right\} = B \begin{Bmatrix} Se_l\,(v)\; Je_l\,(\zeta) \\ So_l\,(v)\; Jo_l\,(\zeta) \end{Bmatrix} e^{j\omega t - \gamma x} = B\,\{Se_l\,(v)\; Je_l\,(\zeta) + p\, So_l(v)\; Jo_l(\zeta)\}\, e^{j\omega t - \gamma x}$$

where B_1, B_2, p and B are constants.

It is easy hereafter to present the solutions for the TM and TE waves. They are two mutually independent solutions. According to definition, $H_x = 0$ for the TM wave and $E_x = 0$ for the TE wave. The desired values are obtained directly from the expressions for E and H in component form presented above.

Appendix 2

We follow here the method of notation employed by Stratton, Morse, Chu and Hutner in their book »Elliptic Cylinder and Spheroidal Wave Functions». In principle the wave problems can be solved with as equal ease in elliptic coordinates as in rectangular, circular cylinder, or spheroidal coordinates. Since, however, several different methods of notation are being used for Mathieu's functions, it may be appropriate, in order to spare the reader's pains, to collect in the following some important formulas and to present certain defining equations. From the practical viewpoint it has to be regretted that the methods of notation with regard to elliptic coordinates have not yet become standardized in any greater degree.

Mathieu's equation is

$$(z^2 - 1)\ U'' + z\ U' + (c^2 z^2 - b)\ U = 0$$

Its solutions represent angular functions when $|z| \leq 1$ and radial functions when $|z| \geq 1$.

By means of a suitable change of the independent variable, this differential equation can be split up into two equations, one of them applying to the angular functions and the other one to the radial functions. Let us denote the angular functions with S and the radial functions with R. We obtain thus

$$\frac{d^2S}{d\varphi^2} + (b - c^2 \cos^2 \varphi)\ S = 0 \quad ; \quad z = \cos \varphi$$

$$\frac{d^2R}{d\psi^2} + (c^2 \cosh^2 \psi - b)\ R = 0 \quad ; \quad z = \cosh \psi \quad , \quad \psi = j\ \varphi$$

where b is an arbitrary constant of separation

c is a parameter

In physical applications there is the additional requirement that the angular function has to be periodic with regard to the angle. Let b_l denote

that separation constant which gives a periodic solution. Both the angular and the radial functions are further divided into even and odd functions.

The angular function:

The even angular function for a given value b_l is

$$Se_l\,(c,\cos\varphi) = \sum_{n=0,1}^{\infty}{}' D_n^l \cos\,(n\,\varphi)$$

The primed summation is to be extended over even values of n if l is even and over odd values of n if l is odd.

The $D_n^l\,(c)$ values have been tabulated for the eigenvalues of b_l.

The angular function given above is normalized at $\varphi = 0$ so that

$$Se_l\,(c,\,1) = 1 \quad \text{or} \quad \sum_{n=0,1}^{\infty}{}' D_n^l = 1$$

The periodic odd angular functions for given values of $b_l'\,(c)$ are obtained from the expression

$$So_l\,(c,\cos\varphi) = \sum_{n=1,2}^{\infty}{}' F_n^l \sin\,(n\,\varphi)$$

The values of $F_n^l\,(c)$ have been tabulated for the eigenvalues of b_l'.

Normalizing again at $\varphi = 0$, we obtain

$$\sum_{n=1,2}^{\infty}{}' n\,F_n^l = 1 \quad \text{or} \quad \frac{d}{d\varphi}\,So_l\,(c,\cos\varphi) \underset{\varphi\to 0}{\longrightarrow} 1$$

The angular functions given above are angular functions of the first kind. Fortunately, it happens that we need not interest ourselves in angular functions of the second kind in this work, these functions being non-periodic with regard to the angle and, on top of all, having a logarithmic singularity when $\cos\varphi = 0$.

The radial function:

In the case of the radial functions both the radial functions of the first and of the second kind have to be taken into account. Both types of functions are further divided into even and odd functions.

The even radial functions are treated, for the eigenvalues of $b_l(c)$, as follows:

$$Re_l^{(1)}\,\zeta = Je_l\,(c, \cosh\zeta) = \sqrt{\frac{\pi}{2}} \sum_{n=0,1}^{\infty}{}' j^{(l-n)}\, D_n^l\, J_n\,(c \cdot \cosh\zeta)$$

$$Re_l^{(2)}\,\zeta = Ne_l\,(c, \cosh\zeta) = \sqrt{\frac{\pi}{2}} \sum_{n=0,1}^{\infty}{}' j^{(l-n)}\, D_n^l\, N_n\,(c \cdot \cosh\zeta)$$

The upper index 1 refers to functions of the first kind and upper index 2 to functions of the second kind. $J_n(z)$ represents Bessel's functions of the first kind and $N_n(z)$ Bessel's functions of the second kind. It should be noted that for the even radial function of the first kind generally the notation $Je_l(c, z)$ is used and for the corresponding radial function of the second kind the notation $Ne_l(c, z)$.

The odd radial functions are defined as follows:

$$Ro_l^{(1)}(\zeta) = Jo_l(c, \cosh\zeta) = \sqrt{\frac{\pi}{2}} \cdot \tanh\zeta \cdot \sum_{n=1,2}^{\infty}{}' j^{(n-l)}\, n\, F_n^l\, J_n\,(c \cdot \cosh\zeta)$$

$$Ro_l^{(2)}(\zeta) = No_l\,(c, \cosh\zeta) = \sqrt{\frac{\pi}{2}} \cdot \tanh\zeta \cdot \sum_{n=1,2}^{\infty}{}' j^{(n-l)}\, n\, F_n^l\, N_n\,(c \cdot \cosh\zeta)$$

By combination of the functions given above, Mathieu's functions of the third and fourth kind are defined. They are divided into even and odd functions. For even radial functions they are:

$$He_l^{(1)}(c, \cosh\zeta) = Je_l(c, \cosh\zeta) + j\, Ne_l(c, \cosh\zeta)$$

$$He_l^{(2)}(c, \cosh\zeta) = Je_l(c, \cosh\zeta) - j\, Ne_l(c, \cosh\zeta)$$

For odd radial functions there is correspondingly

$$Ho_l^{(1)}(c, \cosh\zeta) = Jo_l(c, \cosh\zeta) + j\, No_l(c, \cosh\zeta)$$

$$Ho_l^{(2)}(c, \cosh\zeta) = Jo_l(c, \cosh\zeta) - j\, No_l(c, \cosh\zeta)$$

The method of notation corresponds to that employed in Hankel's functions.

Since the Wronskian of the radial function has to be known in this work, it shall be written down, too:

$$Je_l\,(c,\,\cosh\zeta)\,\frac{d}{d\zeta}\,Ne_l\,(c,\,\cosh\zeta) - Ne_l\,(c,\,\cosh\zeta)\,\frac{d}{d\zeta}\,Je_l\,(c,\,\cosh\zeta) = 1$$

$$Jo_l\,(c,\,\cosh\zeta)\,\frac{d}{d\zeta}\,No_l\,(c,\,\cosh\zeta) - No_l\,(c,\,\cosh\zeta)\,\frac{d}{d\zeta}\,Jo_l\,(c,\,\cosh\zeta) = 1$$

Appendix 3

According to the notations introduced in the work, there is

$$k_{01} = \frac{\omega}{v_{01}} \quad ; \quad z_{01} = \frac{U_{01}}{J_{01}}$$

$$k_{01} = \frac{\omega}{v_{02}} \quad ; \quad z_{02} = \frac{U_{01}}{J_{02}}$$

Consequently,

$$\frac{k_{01}}{z_{01}} = \frac{\omega}{v_{01}} \cdot \frac{J_{01}}{U_{01}} \quad \text{and} \quad \frac{k_{02}}{z_{02}} = \frac{\omega}{v_{02}} \cdot \frac{J_{02}}{U_{02}}$$

According to Child-Langmuir's space charge law there is at a fixed distance

$$\begin{aligned} J_{01} &= \varkappa \cdot U_{01}^{3/2} \\ J_{02} &= \varkappa \cdot U_{02}^{3/2} \end{aligned} \qquad \text{where } \varkappa = \text{constant}$$

Hence

$$\frac{k_{01}}{z_{01}} = \frac{\omega}{v_{01}} \cdot \varkappa \cdot U_{01}^{1/2} = \varkappa \cdot \omega \cdot \frac{1}{\sqrt{2\frac{e}{m}}} \cdot \frac{U_{01}^{1/2}}{U_{01}^{1/2}} = \varkappa \frac{\omega}{\sqrt{2\frac{e}{m}}}$$

It follows that

$$\frac{k_{01}}{z_{01}} = \frac{k_{02}}{z_{01}} = \frac{k_0}{z_0}$$

It can be seen that the ratio $\frac{k_{0n}}{z_{0n}}$ is independent of the voltage U_{0n}.

Appendix 4

In the asymmetric case, obviously, the system of equations corresponding to equations (70) is found to be

$$\left.\begin{aligned} &\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+\frac{u}{d}=B\\ &\frac{x}{a^2}+\frac{y}{b^2}+\frac{z}{c^2}+\frac{u}{d^2}=0\\ &\frac{x}{b}+\frac{y}{a}+\frac{z}{d}+\frac{u}{c}=0\\ &\frac{x}{b^2}+\frac{y}{a^2}+\frac{z}{d^2}+\frac{u}{c^2}=0 \end{aligned}\right\} \quad \therefore \quad \left\{\begin{aligned} &\alpha\ x+\beta\ y+\gamma\ z+\delta\ u=B\\ &\alpha^2 x+\beta^2 y+\gamma^2 z+\delta^2 u=0\\ &\beta\ x+\alpha\ y+\delta\ z+\gamma\ u=0\\ &\beta^2 x+\alpha^2 y+\delta^2 z+\gamma^2 u=0 \end{aligned}\right.$$

where

$$\alpha=\frac{1}{a}\ ,\ \beta=\frac{1}{b}\ ,\ \gamma=\frac{1}{c}\quad\text{and}\quad \delta=\frac{1}{d}$$

This system is easy to solve. To this purpose it is written

$$\left.\begin{aligned} &(\alpha+\beta)\ (x+y)+(\gamma+\delta)\ (z+u)=B\\ &(\alpha-\beta)\ (x-y)+(\gamma-\delta)\ (z-u)=B\\ &(\alpha^2+\beta^2)\ (x+y)+(\gamma^2+\delta^2)\ (z+u)=0\\ &(\alpha^2-\beta^2)\ (x-y)+(\gamma^2-\delta^2)\ (z-u)=0 \end{aligned}\right\} \quad \text{Let be} \quad \left\{\begin{aligned} &x+y=x_1\\ &x-y=x_2\\ &z+u=z_1\\ &z-u=z_2 \end{aligned}\right.$$

Hence

$$\left.\begin{aligned} &(\alpha+\beta)\ x_1+(\gamma+\delta)\ z_1=B\\ &(\alpha-\beta)\ x_2+(\gamma-\delta)\ z_2=B\\ &(\alpha^2+\beta^2)\ x_1+(\gamma^2+\delta^2)\ z_1=0\\ &(\alpha^2-\beta^2)\ x_2+(\gamma^2-\delta^2)\ z_2=0 \end{aligned}\right\} \quad \therefore \quad \left\{\begin{aligned} &x_{\frac{1}{2}}=\frac{(\gamma^2\pm\delta^2)\ B}{(\alpha\pm\beta)\ (\gamma^2\pm\delta^2)-(\gamma\pm\delta)\ (\alpha^2\pm\beta^2)}\\ &z_{\frac{1}{2}}=\frac{(\alpha^2\pm\beta^2)\ B}{(\gamma\pm\delta)\ (\alpha^2\pm\beta^2)-(\alpha\pm\beta)\ (\gamma^2\pm\delta^2)} \end{aligned}\right.$$

Therefore

$$x = +\frac{1}{2} B \left[\frac{\gamma^2+\delta^2}{(\alpha+\beta)(\gamma^2+\delta^2)-(\gamma+\delta)(\alpha^2+\beta^2)} + \frac{\gamma+\delta}{(\alpha-\beta)(\gamma+\delta-\alpha-\beta)}\right]$$

$$y = +\frac{1}{2} B \left[\frac{\gamma^2+\delta^2}{(\alpha+\beta)(\gamma^2+\delta^2)-(\gamma+\delta)(\alpha^2+\beta^2)} - \frac{\gamma+\delta}{(\alpha-\beta)(\gamma+\delta-\alpha-\beta)}\right]$$

$$z = +\frac{1}{2} B \left[\frac{\alpha^2+\beta^2}{(\gamma+\delta)(\alpha^2+\beta^2)-(\alpha+\beta)(\gamma^2+\delta^2)} + \frac{\alpha+\beta}{(\gamma-\delta)(\alpha+\beta-\gamma-\delta)}\right]$$

$$u = +\frac{1}{2} B \left[\frac{\alpha^2+\beta^2}{(\gamma+\delta)(\alpha^2+\beta^2)-(\alpha+\beta)(\gamma^2+\delta^2)} - \frac{\alpha+\beta}{(\gamma-\delta)(\alpha+\beta-\gamma-\delta)}\right]$$

Obviously, thus, we may write

$$x = B_{n1} = B'_{n1} + B''_{n1}$$

$$y = B_{n2} = B'_{n1} - B''_{n1}$$

$$z = B_{n3} = B'_{n3} + B''_{n3}$$

$$u = B_{n4} = B'_{n3} - B''_{n3}$$

Considering the employed notations, we obtain

$$B_{n1} = \frac{1}{2} B \frac{\dfrac{c^2+d^2}{c^2\,d^2}}{\dfrac{a+b}{a\,b}\cdot\dfrac{c^2+d^2}{c^2\,d^2} - \dfrac{c+d}{c\,d}\cdot\dfrac{a^2+b^2}{a^2\,b^2}} = \frac{1}{2} B \frac{a\,b\,c\,d}{(a+b)(c\,d-a\,b)}$$

$$B''_{n1} = \frac{1}{2} B \frac{a\,b}{(b-a)\left(1-\dfrac{c\,d}{a\,b}\right)} = \frac{1}{2} B \frac{\delta_n (1+\sqrt{1+4\,w_n^2})^2}{4\sqrt{(1+4\,w_n^2)}\,[\sqrt{1+4\,w_n^2}+(1+w_n^2)]}$$

$$B'_{n3} = \frac{1}{2} B \frac{\dfrac{a^2+b^2}{a^2\,b^2}}{\dfrac{c+d}{c\,d}\cdot\dfrac{a^2+b^2}{a^2\,b^2} - \dfrac{a+b}{a\,b}\cdot\dfrac{c^2+d^2}{c^2\,d^2}} = \frac{1}{2} B \frac{a\,b\,c\,d}{(a+b)(a\,b-c\,d)}$$

$$B''_{n3} = \frac{1}{2} B \frac{c\,d}{(d-c)\left(1-\dfrac{a\,b}{c\,d}\right)} = j\,\frac{1}{2} B \frac{\delta_n (1-\sqrt{1+4\,w_n^2})^2}{4\sqrt{(1+4\,w_n^2)}\,[\sqrt{1+4\,w_n^2}-(1+w_n^2)]}$$

We find that $B'_{n3} = -B'_{n1} = +\dfrac{1}{2} B \dfrac{\delta_n w_n}{\sqrt{1+4\,w_n^2}}$

It should be noted that

$$\left\{\begin{aligned}
a^2 + b^2 &= +\,2\,\delta_n^2\,[1 + 2\,w_n^2 + \sqrt{1 + 4\,w_n^2}] = \delta_n^2\,(1 + \sqrt{1 + 4\,w_n^2})^2 \\
c^2 + d^2 &= +\,2\,\delta_n^2\,[1 + 2\,w_n^2 - \sqrt{1 + 4\,w_n^2}] = \delta_n^2\,(1 - \sqrt{1 + 4\,w_n^2})^2 \\
a + b &= 2\,\triangle\,k = 2\,\delta_n \cdot w_n = c + d \\
c + d &= 2\,\triangle\,k = 2\,\delta_n \cdot w_n = a + b
\end{aligned}\right.$$

$$\left\{\begin{aligned}
a\,b &= -\,\delta_n^2\,(1 + \sqrt{1 + 4\,w_n^2}) \\
c\,d &= -\,\delta_n^2\,(1 - \sqrt{1 + 4\,w_n^2}) \\
b - a &= -\,2\,\delta_n \sqrt{1 + w_n^2 + \sqrt{1 + 4\,w_n^2}} \\
d - c &= -\,j\,2\,\delta_n \sqrt{\sqrt{1 + 4\,w_n^2} - (1 + w_n^2)}
\end{aligned}\right.$$

The expression of J_{xnT} can now immediately be written in analogy to the expression (75), taking into account the results just obtained.

VALTION TEKNILLINEN TUTKIMUSLAITOS

Julkaisu 37 Publikation

ERRATA

Page	Line	In place of	Read
26	14	$\ldots = Ke_1(\ )$	$\ldots = Ke_1(\ )$, 1 = even and generally Physical quantity $\left[j^{-1} Ne_1 (jc, z) \right] = Re \left[j^{-1} Ne_1 (jc, z) \right] = j^{-1} Ke_1 (c, z)$
77	12	$Bn_1 = \frac{1}{2}B\ldots$	$Bn_1' = \frac{1}{2}B\ldots$

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